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Technical Report No. 4

LINEAR STABILITY OF A TWO-PHASE PROCESS INVOLVING A STEADILY PROPAGATING PLANAR PHASE BOUNDARY IN A SOLID: PART 1. PURELY MECHANICAL CASE

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ABSTRACT

This work investigates the linear stability of an antiplane shear motion which involves a planar phase boundary in an arbitrary element of a wide class of *non-elliptic* generalized neo-Hookean materials which have two distinct elliptic phases. It is shown, via a ***normal mode analysis***, that, in the absence of inertial effects, such a process is linearly unstable with respect to a large class of disturbances if and only if the ***kinetic response function***—a constitutively supplied entity which gives the normal velocity of a phase boundary in terms of the driving traction which acts on it or ***vice versa***—is locally decreasing as a function of the appropriate argument. An alternate analysis, in which the linear stability problem is recast as a functional equation for the interface position, allows the interface to be tracked subsequent to perturbation. A particular choice of the initial disturbance is used to show that, in the case of an unstable response, the morphological character of the phase boundary evolves to qualitatively resemble the ***plate-like*** structures which are found in displacive solid-solid phase transformations. In the presence of inertial effects a combination of normal mode and energy analyses are used to show that the condition which is necessary and sufficient for instability with respect to the relevant class of perturbations in the absence of inertia remains necessary for the entire class of perturbations and sufficient for all but a very special, and physically unrealistic, subclass of these perturbations. The linear stability of the relevant process depends, therefore, entirely upon the transformation kinetics intrinsic to the kinetic response function.

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1. INTRODUCTION

Displacive solid-solid phase transformations occur in a wide variety of metallic and ceramic alloys. The different phases of a material capable of undergoing such a transformation are generally distinguished by distinct crystal structures. Transformations involving such materials are characterized by the existence of interfaces, or *phase boundaries*, which segregate material in different phases. Of particular interest is the growth stage of a displacive solid-solid phase transformation which is directly related to the kinetics governing the motion of phase boundaries. Some experimental work directed at understanding the growth stage of these transformations has been performed. NISHIYAMA [25] has separated the transformation kinetics of relevant materials into three classes based upon the speed at which they occur. Depending on how they are loaded, some materials may exhibit kinetics which fall into any or all of these three classes. In the fastest of these, phase boundaries propagate at velocities which are of the same order of magnitude as the velocity of shear wave propagation; in the remaining two classes the velocities with which phase boundaries propagate are many orders of magnitude smaller the the shear wave speed. The work of GRUJICIC, OLSON & OWEN [16] and CLAPP & YU [10] suggests that slowly propagating phase boundaries are most often observed to be planar in structure, while those which propagate rapidly often display highly complex geometries involving *plate-like* or *dendritic* structures. These complicated structures are reminiscent of those which occur in crystal growth, that are known to evolve from states involving planar interfaces which separate solidified crystal material from liquid melt.[1] It is, therefore, natural to speculate as to whether the complicated plate-like morpholgies observed in rapid displacive solid-solid phase transformations can emerge in an analogous fashion from their slow counterparts.

Finite elastic dynamical processes in materials capable of sustaining equilibria with discontinuous deformation gradients have figured prominently in recent

[1] LANGER [22] provides an overview of such phenomena.

continuum mechanical treatments of displacive solid-solid phase transformations.[2] In a ***homogeneous, hyperelastic*** material equilibrium states with discontinuous deformation gradients occur only if the relevant ***elastic potential*** allows for a loss of ellipticity—at certain values of the deformation gradient—in the associated displacement equations of ***equilibrium.***[3] Materials characterized by elastic potentials which allow such a loss of ellipticity are referred to as ***non-elliptic.*** Of particular importance in most of the work that has been done in this area are non-elliptic materials which have two disjoint elliptic phases. Such hypothetical materials serve as models for actual materials which can sustain displacive solid-solid phase transformations; surfaces which, in either equilibrium or dynamics, separate the different phases of a non-elliptic material function as models for the phase boundaries which occur in actual materials and, hence, are referred to as such.

Despite the apparent dearth of experimental information regarding the issue of whether the growth stage of displacive solid-solid phase transformations can involve the emergence of complicated dendritic structures from planar phase boundaries, it is legitimate to examine this topic from an analytical perspective in the foregoing continuum mechanical context. Except for the work of SILLING [30], the bulk of the continuum mechanical investigations which consider dynamical processes are confined to one-dimensional bar theory and, hence, are not of direct bearing on the issue of phase boundary morphology. SILLING [30] has demonstrated, through an asymptotic analysis, that a particular ***generalized neo-Hookean*** material is capable of sustaining a motion which involves a steadily propagating cusped surface of discontinuity which segregates distinct elliptic phases of the relevant material. This cusped phase boundary can be thought of as a model for one which would accompany a single plate-like structure in an actual displacive solid-solid phase transformation. SILLING [30] also performs numerical

[2] See, *e.g.*, ABEYARATNE & KNOWLES [4–6], JAMES [18], PENCE [27] and SILLING [30].

[3] For a discussion of this issue consult, for instance, ROSAKIS [28].

calculations which seem to support his asymptotic results. It is important to note that this work does not consider the issue of the emergence of the cusped phase boundary from a planar one.

In analogy to the large body of analytical work which has been directed at modeling the emergence of dendritic structures from planar interfaces in the process of crystal growth,[4] it seems reasonable—as a first step in addressing the issue at hand—to investigate the stability of a two-phase process involving a steadily propagating planar phase boundary in a non-elliptic material. In a study which focuses primarily on contructively establishing the existence of two-phase equilibria in a special non-elliptic generalized neo-Hookean material FRIED [14] also analyzes—in an *inertia-free* setting—the stability of such a state with respect to a particular class of perturbations.[5] Together, the narrow class of perturbations which is considered, the absence of inertial effects, and the constitutive specialization which is adhered to severely restrict the generality of the results which are obtained in [14]. The objective of the present inquiry is, therefore, to perform a more general stability analysis where a two-phase process involving a steadily propagating planar phase boundary in a wide class of non-elliptic generalized neo-Hookean materials is subjected to a broad class of disturbances and inertial effects are taken into consideration. It will transpire, however, that the stability results which are obtained are consistent with those secured by FRIED [14].

Chapter 2 is devoted to preliminaries. After a brief overview of the notation to be used, Section 2.1 introduces the kinematics and fundamental balance principles which will be needed in the following. Section 2.2 explains the constitutive restrictions which will be adhered to throughout this work. Section 2.3 is concerned with the notions of *mechanical dissipation* and *driving traction* which are associated with phase boundaries; these lead naturally to the consideration of a *kinetic relation*—which gives the normal velocity of a phase boundary in terms

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[4] See, for example, LANGER [22], MULLINS & SEKERKA [24] and STRAIN [31].

[5] The relevant material is also that used by SILLING in [29–30].

of the driving traction which acts on it or *vice versa*—and an associated *kinetic response function*. In the final section of Chapter 2, the kinematics are specialized to those of *antiplane shear*.

Chapter 3 concentrates upon a linear stability analyis of a two-phase process involving a steadily propagating planar phase boundary in a non-elliptic generalized neo-Hookean material which obeys the Baker-Ericksen inequality. The process to be perturbed, which involves an antiplane shear deformation, is introduced in Section 3.1. In Section 3.2 the class of perturbations which will be applied to the base process are then introduced. Each admissible perturbation involves, in general, a disturbance of the configuration of the phase boundary and of the displacement and velocity fields in a small neighborhood of the phase boundary—all of which are assumed to be small in some appropriate sense. The kinematics of the perturbation are also restricted to those of antiplane shear. It is assumed, furthermore, that the post-perturbation deformation remains an antiplane shear and involves only one phase boundary. Section 3.3 is devoted to the linearization about the base process of the field equations, which hold away from the phase boundary, about the base process introduced in Section 3.1. In a similar manner, Section 3.4 is concerned with the linearization about the base process of the jump conditions and kinetic relation which hold on the phase boundary. A summary of the complete linearized system of field equations, jump conditions, kinetic relation, boundary and initial conditions which describe the process generated by the perturbation is presented in Section 3.5. Included are both the inertial and inertia-free cases. In Section 3.6 a *normal mode analysis* is performed in the absence of inertial effects. A condition necessary and sufficient for the base process to be unstable with respect to any perturbation of the type introduced in Section 3.2 is obtained. This condition involves only the local behavior of the derivative of the (essentially arbitrary) kinetic response function introduced in Section 2.3. An alternative to the normal mode analysis of Section 3.6 is performed in Section 3.7. Here the relevant initial boundary value problem is converted into a functional

initial value problem for the correction to the interface position arising from the perturbation. Analysis of this problem yields identical stability criteria to those which are obtained in Section 3.6. Moreover, when instability is present, its manifestation can be tracked over a finite time interval upon which the linearization performed in the foregoing remains valid. The results suggest the emergence of plate-like or dendritic structures. Section 3.8 contains both a normal mode and an energy analyis in the case where inertial effects are included. First, the normal mode analysis leads to a necessary and sufficient condition for the base process to be unstable with respect to a special subset of the class of perturbations introduced in Section 3.2. The operative stability criterion is identical to that which holds for all initial disturbances in the inertia-free case. An argument based upon a Fourier-Laplace transform analysis of the relevant initial boundary value problem is then used to show that all but a very special subset of the full class of perturbations introduced in Section 3.2 are covered by the normal mode analysis. Section 3.8 is concluded with an energy analysis which is used to show that the sufficient condition referred to above is also necessary for the base process to be unstable with respect to any perturbation of the class introduced in Section 3.2. Hence, it is shown that the presence of inertia does not qualitatively alter the linear stability of the base process of interest. Finally, in Section 3.9, a discussion which focusses on the physical reasonableness of admissible non-monotonic kinetic response functions is undertaken.

2. PRELIMINARIES

2.1. Notation, kinematics and balance principles. In the following $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The intervals $(0, \infty)$ and $[0, \infty)$ are represented by $\mathbb{R}_+$ and $\bar{\mathbb{R}}_+$. The symbol $\mathbb{R}^n$, with n equal to 2 or 3, represents real n-dimensional space equipped with the standard Euclidean norm. If U is a set, then its closure, interior and boundary are designated by $\bar{U}$, $\mathring{U}$, and ∂U, respectively. The complement of a set V with respect to U is written as $U \setminus V$. Given a function $\psi : U \to W$ and a subset V of U, $\psi(V)$ stands for the image of V under the map ψ.

Vectors and linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ (referred to herein as *tensors*) are distinguished from scalars with the aid of boldface type—lower and upper case for vectors and tensors, respectively. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$, their inner product is then written as $\mathbf{a} \cdot \mathbf{b}$; the Euclidean norm of $\mathbf{a}$ is, further, written as $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The set of unit vectors—that is, vectors with unit Euclidean norm—in $\mathbb{R}^3$ is designated by $\mathcal{N}$. The symbol $\mathcal{L}$ refers to the set of tensors, $\mathcal{L}_+$ denotes the set of all tensors with positive determinant, and $\overset{+}{\mathcal{S}}$ stands for the collection of all symmetric positive definite tensors. If $\mathbf{F}$ is in $\mathcal{L}$ then $\mathbf{F}^T$ represents its transpose; if, moreover, $\det \mathbf{F} \neq 0$, then the inverse of $\mathbf{F}$ and its transpose are written as $\mathbf{F}^{-1}$ and $\mathbf{F}^{-T}$, respectively. The notation $\mathbf{a} \otimes \mathbf{b}$ refers to the tensor $\mathbf{A}$, formed by the outer product of $\mathbf{a}$ with $\mathbf{b}$, defined such that $\mathbf{A}\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for any vector $\mathbf{c}$ in $\mathbb{R}^3$. If $\mathbf{A}$ and $\mathbf{B}$ are tensors then their inner product is written as $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr} \mathbf{A}\mathbf{B}^T$.

When component notation is used, Greek indices range only over $\{1, 2\}$; summation of repeated indices over the appropriate range is implicit. A subscript preceded by a comma denotes partial differentiation with respect to the corresponding coordinate. Also, a superposed dot signifies partial differentiation with respect to time.

Consider, now, a body $\mathcal{B}$ which, in a reference configuration, occupies a region $\mathcal{R}$ contained in $\mathbb{R}^3$. A motion of $\mathcal{B}$ on a time interval $\mathcal{T} \subset \mathbb{R}$ is characterized by

a one-parameter family of invertible mappings $\hat{\mathbf{y}}(\cdot,t):\mathcal{R}\to\mathcal{R}_t$, with

$$\hat{\mathbf{y}}(\mathbf{x},t)=\mathbf{x}+\mathbf{u}(\mathbf{x},t)\quad\forall(\mathbf{x},t)\in\mathcal{M}, \tag{2.1.1}$$

where $\mathcal{M}=\mathcal{R}\times\mathcal{T}$ represents the *trajectory* of the motion. Assume that the *deformation* $\hat{\mathbf{y}}$, or equivalently the *displacement* $\mathbf{u}$, is continuous and possesses piecewise continuous first and second partial derivatives on $\mathcal{M}$. Let S_t be the set of points contained in $\mathcal{R}$ defined so that, at each instant t in $\mathcal{T}$, $\hat{\mathbf{y}}(\cdot,t)$ is twice continuously differentiable on the set $\mathcal{R}\setminus S_t$. Let the set Σ be defined by

$$\Sigma=\{(\mathbf{x},t)\,|\,\mathbf{x}\in S_t,t\in\mathcal{T}\}. \tag{2.1.2}$$

Introduce the *deformation gradient tensor* $\mathbf{F}:\mathcal{M}\setminus\Sigma\to\mathcal{L}$ by

$$\mathbf{F}(\mathbf{x},t)=\nabla\hat{\mathbf{y}}(\mathbf{x},t)\quad\forall(\mathbf{x},t)\in\mathcal{M}\setminus\Sigma, \tag{2.1.3}$$

where the associated *Jacobian determinant*, $J:\mathcal{M}\setminus\Sigma\to I\!R$, of $\hat{\mathbf{y}}$ is restricted to be strictly positive on its domain of definition:

$$J(\mathbf{x},t)=\det\mathbf{F}(\mathbf{x},t)>0\quad\forall(\mathbf{x},t)\in\mathcal{M}\setminus\Sigma.$$

Hence, $\mathbf{F}:\mathcal{M}\setminus\Sigma\to\mathcal{L}_+$. The *left Cauchy-Green tensor* $\mathbf{G}:\mathcal{M}\setminus\Sigma\to\overset{+}{\mathcal{S}}$ corresponding to the deformation $\hat{\mathbf{y}}$ is given by

$$\mathbf{G}(\mathbf{x},t)=\mathbf{F}(\mathbf{x},t)\mathbf{F}^T(\mathbf{x},t)\quad\forall(\mathbf{x},t)\in\mathcal{M}\setminus\Sigma. \tag{2.1.4}$$

The *deformation invariants* associated with $\hat{\mathbf{y}}$ exist on $\mathcal{M}\setminus\Sigma$ and are supplied through the fundamental scalar invariants of $\mathbf{G}$:

$$I_1(\mathbf{G})=\operatorname{tr}\mathbf{G},\quad I_2(\mathbf{G})=\tfrac{1}{2}\left((\operatorname{tr}\mathbf{G})^2-\operatorname{tr}(\mathbf{G}^2)\right),\quad I_3(\mathbf{G})=\det\mathbf{G}. \tag{2.1.5}$$

With the above kinematic antecedents in place introduce the *nominal mass density* $\rho : \mathcal{R} \to \mathbb{R}_+$, the *nominal body force per unit mass* $\mathbf{b} : \mathcal{M} \to \mathbb{R}^3$, and the *nominal stress tensor* $\mathbf{S} : \mathcal{M} \setminus \Sigma \to \mathcal{L}$, and suppose that ρ is constant on $\mathcal{R}$ and $\mathbf{b}$ is continuous on $\mathcal{M}$, while $\mathbf{S}$ is piecewise continuous on $\mathcal{M}$, continuous on $\mathcal{M} \setminus \Sigma$, and has a piecewise continuous gradient on $\mathcal{M}$. Let ρ_* be the mass density in the deformed configuration associated with $\hat{\mathbf{y}}$. Given a *regular* subregion $\mathcal{P}$ of $\mathcal{R}$, with $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ for each t in $\mathcal{T}$, let $\mathbf{m} : \partial\mathcal{P} \to \mathcal{N}$ denote the unit outward normal to $\partial\mathcal{P}$. Then the global balance laws of mass, linear momentum, and angular momentum require that

$$\int_{\mathcal{P}} \rho\, dV = \int_{\hat{\mathbf{y}}(\mathcal{P})} \rho_*\, dV \quad \text{on} \quad \mathcal{T}, \tag{2.1.6}$$

$$\int_{\partial\mathcal{P}} \mathbf{S}\mathbf{m}\, dA + \int_{\mathcal{P}} \rho\mathbf{b}\, dV = \overline{\int_{\mathcal{P}} \rho\dot{\mathbf{u}}\, dV}^{\,\boldsymbol{\cdot}} \quad \text{on} \quad \mathcal{T}, \tag{2.1.7}$$

and

$$\int_{\partial\mathcal{P}} \hat{\mathbf{y}} \wedge \mathbf{S}\mathbf{m}\, dA + \int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\mathbf{b}\, dV = \overline{\int_{\mathcal{P}} \hat{\mathbf{y}} \wedge \rho\dot{\mathbf{u}}\, dV}^{\,\boldsymbol{\cdot}} \quad \text{on} \quad \mathcal{T}, \tag{2.1.8}$$

respectively, for every such regular subregion $\mathcal{P}$ contained in $\mathcal{R}$.

Localization of the balance laws (2.1.6)–(2.1.8) at an arbitrary point contained in the interior of $\mathcal{M} \setminus \Sigma$ yields the following familiar field equations:

$$\begin{aligned} \rho &= \rho_*(\hat{\mathbf{y}})J \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \\ \nabla \cdot \mathbf{S} + \rho\mathbf{b} &= \rho\ddot{\mathbf{u}} \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \\ \mathbf{S}\mathbf{F}^T &= \mathbf{F}\mathbf{S}^T \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \end{aligned} \tag{2.1.9}$$

Suppose, from now on, that the set S_t is a regular surface for every t in $\mathcal{T}$. The set Σ then represents the trajectory of a surface of discontinuity in $\mathbf{F}$ and $\mathbf{S}$. Let $g(\cdot, t)$ denote a generic field quantity $g(\cdot, t) : S_t \to \mathbb{R}$ which is discontinuous

across S_t at the instant t in $\mathcal{T}$. Define the *jump* $[\![g(\cdot,t)]\!]$ of $g(\cdot,t)$ across S_t by

$$[\![g(\mathbf{x},t)]\!] = \lim_{h\searrow 0}\big(g\big(\mathbf{x}+h\mathbf{n}(\mathbf{x},t),t\big) - g\big(\mathbf{x}-h\mathbf{n}(\mathbf{x},t),t\big)\big) \quad \forall(\mathbf{x},t)\in\Sigma, \quad (2.1.10)$$

where $\mathbf{n}(\cdot,t) : S_t \to \mathcal{N}$ is a unit normal to S_t at each t in $\mathcal{T}$. Then, localization of (2.1.6)–(2.1.8) at an arbitrary point in Σ yields the following jump conditions

$$\begin{aligned} [\![\rho_*(\hat{\mathbf{y}})J]\!] &= 0 \quad \text{on} \quad \Sigma, \\ [\![\mathbf{Sn}]\!] + \rho V_n[\![\dot{\mathbf{u}}]\!] &= \mathbf{0} \quad \text{on} \quad \Sigma, \end{aligned} \qquad (2.1.11)$$

where $V_n(\cdot,t) : S_t \to \mathbb{R}$ denotes the component of the velocity of the surface S_t in the direction of $\mathbf{n}(\cdot,t)$ at the instant t in $\mathcal{T}$.

Equations $(2.1.9)_1$ and $(2.1.11)_1$ are, evidently, completely decoupled from equations $(2.1.9)_{2,3}$ and $(2.1.11)_2$; that is, given a solution to, say, a boundary value problem involving $(2.1.9)_{2,3}$ and $(2.1.10)_2$, ρ_* can be calculated *a posteriori*. For this reason equations $(2.1.9)_1$ and $(2.1.11)_1$ will be disregarded in the subsequent analysis.

In this investigation an *inertia-free* motion is defined as one wherein the inertial terms on the right hand sides of the global balance equations (2.1.7) and (2.1.8) are replaced by the zero vector. In the context of an inertia-free motion the field equation $(2.1.9)_2$ simplifies to read

$$\nabla\cdot\mathbf{S} + \rho\mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathcal{M}\setminus\Sigma, \qquad (2.1.12)$$

and the jump condition $(2.1.11)_2$ becomes

$$[\![\mathbf{Sn}]\!] = \mathbf{0} \quad \text{on} \quad \Sigma. \qquad (2.1.13)$$

Equations $(2.1.9)_{1,3}$ and $(2.1.11)_1$ remain, of course, unaltered.

In addition to the jump conditions given in (2.1.11) in the inertial case or $(2.1.10)_1$ and (2.1.13) in the inertia-free case, the stipulated continuity of $\hat{\mathbf{y}}$ gives the following *kinematic* jump condition

$$[\![\mathbf{u}]\!] = \mathbf{0} \quad \text{on} \quad \Sigma. \tag{2.1.14}$$

2.2. Constitutive assumptions. Let $\mathcal{B}$ be composed of a hyperelastic material which is homogeneous, isotropic and incompressible. Since $\mathcal{B}$ is hyperelastic its mechanical response is governed by an elastic potential or *strain energy per unit reference volume*. The homogeneity of $\mathcal{B}$ implies that the elastic potential does not depend explicitly on position in the reference configuration. Furthermore, because $\mathcal{B}$ is isotropic the elastic potential can depend on the deformation gradient $\mathbf{F}$ only through the deformation invariants $I_k(\mathbf{G})$ defined in (2.1.5). The incompressibility of $\mathcal{B}$ requires that the deformation $\hat{\mathbf{y}}$ be *isochoric*, *i.e.*,

$$I_3\big(\mathbf{G}(\mathbf{x},t)\big) = J^2(\mathbf{x},t) = 1 \quad \forall(\mathbf{x},t) \in \mathcal{M} \setminus \Sigma. \tag{2.2.1}$$

An additional consequence of isotropy is, therefore, that the elastic potential can be expressed as a function solely of the first two deformation invariants. It can also be demonstrated via (2.1.5) that, when (2.2.1) holds, $I_\alpha(\mathbf{G}(\mathbf{x},t)) \geq 3$ for all $(\mathbf{x},t)$ contained in $\mathcal{M} \setminus \Sigma$. Now, let $\tilde{W} : [3,\infty) \times [3,\infty) \to I\!R$ denote an elastic potential which characterizes $\mathcal{B}$ and assume that $\tilde{W}$ is continuously differentiable with piecewise continuous second derivatives on its domain of definition. The nominal stress response of $\mathcal{B}$ is then determined through $\tilde{W}$ up to an arbitrary pressure $p : \mathcal{M} \setminus \Sigma \to I\!R$ required to accomodate the kinematic constraint (2.2.1) imposed by the incompressibility of $\mathcal{B}$: *viz.*,

$$\mathbf{S} = 2\left(\tilde{W}_{I_1}(I)\mathbf{F} + \tilde{W}_{I_2}(I)\big(I_1(\mathbf{G})\mathbf{1} - \mathbf{G}\big)\mathbf{F}\right) - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{M} \setminus \Sigma, \tag{2.2.2}$$

where $I : \mathcal{M} \setminus \Sigma \to [3,\infty) \times [3,\infty)$ is given by

$$I(\mathbf{x},t) = \big(I_1\big(\mathbf{G}(\mathbf{x},t)\big), I_2\big(\mathbf{G}(\mathbf{x},t)\big)\big) \quad \forall(\mathbf{x},t) \in \mathcal{M} \setminus \Sigma.$$

Following GURTIN [17], let the class of ***generalized neo-Hookean*** materials refer to that subset of hyperelastic materials, first introduced by KNOWLES [19], which are homogeneous, isotropic and incompressible with elastic potential independent of the second deformation invariant $(2.1.5)_2$. Assume, henceforth, that $\mathcal{B}$ is composed of a generalized neo-Hookean material with elastic potential $W : [3,\infty) \rightarrow \mathbb{R}$, where W is continuously differentiable with piecewise continuous derivative on $[3,\infty)$. Then, by (2.2.2), the nominal stress response of $\mathcal{B}$ is determined by

$$\mathbf{S} = 2W'(I_1(\mathbf{G}))\mathbf{F} - p\mathbf{F}^{-T} \quad \text{on} \quad \mathcal{M} \setminus \Sigma. \tag{2.2.3}$$

Suppose also that the elastic potential is ***normalized*** so that

$$W(3) = 0. \tag{2.2.4}$$

Choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and consider the response of the material at hand to a simple shear deformation $\hat{\mathbf{y}}$ given by

$$\hat{\mathbf{y}}(\mathbf{x},t) = (\mathbf{1} + \gamma \mathbf{e}_3 \otimes \mathbf{e}_1)\mathbf{x} \quad \forall (\mathbf{x},t) \in \mathcal{M}, \tag{2.2.5}$$

where the constant γ—assumed non-negative without loss of generality—denotes the amount of shear. From (2.1.3), (2.2.3) and (2.2.5) the nominal shear stress corresponding to the deformation $\hat{\mathbf{y}}$ is, for each γ in $\bar{\mathbb{R}}_+$, found to be

$$\mathbf{e}_3 \cdot \mathbf{S}\mathbf{e}_1 = 2\gamma W'(3+\gamma^2) =: \tau(\gamma). \tag{2.2.6}$$

In [19–20] KNOWLES demonstrates that the 31 and 32 components of nominal and Cauchy shear stress are, in the present setting, equal. The function $\tau : \bar{\mathbb{R}}_+ \rightarrow \mathbb{R}$ is, hence, referred to as the ***shear stress response function*** of the generalized neo-Hookean material, characterized by W, in simple shear. An immediate consequence of (2.2.4) and (2.2.6) is

$$W(I_1) = \int_0^{\sqrt{I_1 - 3}} \tau(\gamma)\, d\gamma \quad \forall I_1 \in [3,\infty), \tag{2.2.7}$$

so that the response of a generalized neo-Hookean material, in ***all*** three dimensional deformations, is, up to a hydrostatic pressure, completely characterized by specifying the shear stress response function τ. Define the *secant modulus in shear* $M : \bar{\mathbb{R}}_+ \to \mathbb{R}$ of a generalized neo-Hookean material with elastic potential W by

$$M(\gamma) = 2W'(3 + \gamma^2) \quad \forall \gamma \in \bar{\mathbb{R}}_+, \tag{2.2.8}$$

and assume that, in compliance with the *Baker-Ericksen inequality*,

$$M(\gamma) > 0 \quad \forall \gamma \in \mathbb{R}_+. \tag{2.2.9}$$

Assume, also, that $M(0) > 0$ so that the infinitesimal shear modulus of the material at hand is positive. Note from (2.2.6) and (2.2.8) that the shear stress response function τ must also satisfy

$$\tau(0) = 0, \quad \tau'(0) = M(0). \tag{2.2.10}$$

Observe, in addition, that the stipulated smoothness of W guarantees that both τ and M are piecewise continuously differentiable on $\bar{\mathbb{R}}_+$.

Despite the significant restrictions which have been placed upon the class of materials which will be considered in this investigation, no presuppositions have been made regarding the sign of the derivative—where it exists—of the shear stress response function corresponding to the generalized neo-Hookean material defined through (2.2.6). In [20] KNOWLES shows that the monotonicity of the shear stress response function τ is related directly to the ellipticity of the generalized neo-Hookean material which it characterizes: if τ is not a monotonically increasing function on its domain of definition then the associated material is non-elliptic. This investigation will make use of a particular subclass of non-elliptic generalized neo-Hookean materials, first suggested by ABEYARATNE [1]; this class of materials is characterized by the set of shear stress response functions τ which

are continuous on $\bar{\mathbb{R}}_+$ and piecewise continuously differentiable on $\bar{\mathbb{R}}_+ \setminus \{\underset{*}{\gamma}, \overset{*}{\gamma}\}$, where $0 < \underset{*}{\gamma} < \overset{*}{\gamma}$, such that

$$\begin{aligned} \tau' > 0 \quad &\text{on} \quad \bar{\mathbb{R}}_+ \setminus [\underset{*}{\gamma}, \overset{*}{\gamma}], \\ \tau' < 0 \quad &\text{on} \quad (\underset{*}{\gamma}, \overset{*}{\gamma}). \end{aligned} \tag{2.2.11}$$

The sets of shear strains lying in the intervals $[0, \underset{*}{\gamma})$ and $(\overset{*}{\gamma}, \infty)$ are referred to as the ***high*** and ***low strain phases*** of the generalized neo-Hookean material specified by the shear stress response function τ. Together the high and low strain phases of such a material comprise its ***elliptic phases***. A generalized neo-Hookean material characterized by a shear stress response function of this type will be referred to herein as a ***three-phase*** material. See Figure 1 for a graph of a shear stress response function typical of those which specify three-phase materials.

2.3. Dissipation, driving traction and the kinetic relation. Let $\mathcal{P}$ be a regular subregion contained in $\mathcal{R}$ chosen so that $\partial\mathcal{P} \cap \Sigma$ is a set of measure zero in $\partial\mathcal{P}$. The total mechanical energy contained in $\mathcal{P}$ at an instant t contained in $\mathcal{T}$ is given, under the present constitutive assumptions, by

$$E(t;\mathcal{P}) = \int_{\mathcal{P}} \left(W\big(I_1(\mathbf{G}(\mathbf{x},t))\big) + \tfrac{1}{2}\rho|\dot{\mathbf{u}}(\mathbf{x},t)|^2\right) dV \quad \forall t \in \mathcal{T}. \tag{2.3.1}$$

A standard calculations then shows, with the aid of (2.2.3), that

$$\dot{E}(t;\mathcal{P}) = \int_{\partial\mathcal{P}} \mathbf{S}(\mathbf{x},t)\mathbf{n}(\mathbf{x},t) \cdot \dot{\mathbf{u}}(\mathbf{x},t)\, dA - \int_{\mathcal{P}\cap S_t} f(\mathbf{x},t)V_n(\mathbf{x},t)\, dA \quad \forall t \in \mathcal{T}, \tag{2.3.2}$$

where $f : \Sigma \to \mathbb{R}$ is the scalar ***driving traction*** given by

$$f(\cdot,t) = [\![W\big(I_1(\mathbf{G}(\cdot,t))\big)]\!] - \langle\!\langle \mathbf{S}(\cdot,t)\rangle\!\rangle \cdot [\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}, \tag{2.3.3}$$

and, given a generic field quantity $g(\cdot,t) : S_t \to \mathbb{R}$ which jumps across S_t at the instant t in $\mathcal{T}$, $\langle\!\langle g(\cdot,t)\rangle\!\rangle$ is defined through

$$\langle\!\langle g(\mathbf{x},t)\rangle\!\rangle = \lim_{h\searrow 0} \tfrac{1}{2}\left(g\big(\mathbf{x} + h\mathbf{n}(\mathbf{x},t),t\big) + g\big(\mathbf{x} - h\mathbf{n}(\mathbf{x},t),t\big)\right) \quad \forall(\mathbf{x},t) \in \Sigma, \tag{2.3.4}$$

In [21] KNOWLES derives (2.3.2) in the absence of inertial effects; following either YATOMI & NISHIMURA [32] or ABEYARATNE & KNOWLES [3] it can be shown that (2.3.3) reduces, in this setting, to

$$f(\cdot,t) = [\![W(I_1(\mathbf{G}(\cdot,t)))]\!] - \overset{\pm}{\mathbf{S}}(\cdot,t)\cdot[\![\mathbf{F}(\cdot,t)]\!] \quad \text{on} \quad S_t \quad \forall t \in \mathcal{T}, \tag{2.3.5}$$

where $\overset{+}{\mathbf{S}}(\cdot,t)$ (*resp.*, $\overset{-}{\mathbf{S}}(\cdot,t)$) is the limiting value of the field $\mathbf{S}(\cdot,t)$ on the side of the interface into which the unit normal $\mathbf{n}(\cdot,t)$ is (*resp.*, is not) directed at the instant t in $\mathcal{T}$.

From (2.3.2) it is clear that the presence of a moving surface of discontinuity S_t of the type considered here may effect the balance of mechanical energy. Let the difference in the rate of work of the mechanical forces external to $\mathcal{P}$ and the rate at which energy is stored in $\mathcal{P}$ be referred to as the *rate of dissipation* of mechanical energy associated with the region $\mathcal{P}$. When treated from a *thermomechanical* perspective, the dissipation rate can be shown to be identical to the product of the temperature and the rate of entropy production—provided that the temperature is spatially uniform and independent of time.[6] The *Clausius-Duhem inequality* then requires that the dissipation rate associated with a motion of the kind envisioned here be non-negative, *i.e.*,

$$\int_{\mathcal{P}\cap S_t} f(\mathbf{x},t)V_n(\mathbf{x},t)\,dA \geq 0 \quad \forall t \in \mathcal{T}, \tag{2.3.6}$$

for every regular subregion $\mathcal{P}$, with $\partial\mathcal{P} \cap S_t$ a set of measure zero in $\partial\mathcal{P}$ at t in $\mathcal{T}$, contained in $\mathcal{R}$. A localization of (2.3.6) at an arbitrary point on the interface therefore yields the inequality

$$fV_n \geq 0 \quad \text{on} \quad \Sigma \tag{2.3.7}$$

as a condition imposed for the *admissibility* of the motion.

[6] For a detailed discussion of the these issues see ABEYARATNE & KNOWLES [3].

Assume that the surface S_t separates, for each t in $\mathcal{T}$, low and high strain elliptic phases of the three phase material at hand. In the context of a motion which involves such an interface it is necessary (see [3–6]) to supplement, in some fashion, the constitutive information which relates the stress and strain fields. An approach to this taken by ABEYARATNE & KNOWLES [3] entails the provision of a *kinetic relation* which gives the normal velocity of the interface in terms of the driving traction that acts on it or *vice versa*. In the former case a constitutive function $\tilde{V} : I\!R \to I\!R$ is provided so that

$$V_n = \tilde{V}(f) \quad \forall f \in I\!R, \tag{2.3.8}$$

while, in the latter case a constitutive function $\tilde{\varphi} : I\!R \to I\!R$ is given so that

$$f = \tilde{\varphi}(V_n) \quad \forall V_n \in I\!R. \tag{2.3.9}$$

The functions $\tilde{V}$ and $\tilde{\varphi}$ are referred to as the *kinetic response functions*. Both varieties of kinetic response functions will be considered in this investigation. If $\tilde{V}$ is such that $\tilde{V}(f)f \geq 0$ for all f in $I\!R$ then (2.3.8) is automatically satisfied and $\tilde{V}$ is referred to as *admissible*. If $\tilde{\varphi}(V)V \geq 0$ for all V in $I\!R$, $\tilde{\varphi}$ is, similarly, referred to as admissible. If an admissible kinetic response function $\tilde{V}$ (or $\tilde{\varphi}$) is continuous on $I\!R$, then it must satisfy $\tilde{V}(0) = 0$ (or $\tilde{\varphi}(0) = 0$). If, in addition, to being admissible, $\tilde{V}$ (or $\tilde{\varphi}$) is continuously differentiable on $I\!R$, then $\tilde{V}'(0) \geq 0$ (or $\tilde{\varphi}'(0) \geq 0$). Otherwise, admissiblility implies nothing with regard to the sign of the derivative of a smooth kinetic response function. All kinetic response functions considered herein are assumed to be admissible. See Figure 2 and Figure 3 for graphs of such kinetic response functions.

ABEYARATNE [1], BALL & JAMES [9], GURTIN [17], FOSDICK & MACSITHIGH [12], and SILLING [29] consider either equilibrium states or inertia-free motions and require that the driving traction, f, be identically equal to zero on Σ. This is equivalent to prescribing a supplementary kinetic relation in the form (2.3.9) with

$\tilde{\varphi}$ identically zero on $I\!R$. Provision of such a kinetic relation is, furthermore, a necessary consequence of requiring that a suitable *energy functional* be minimized at each t in $\mathcal{T}$ (see ABEYARATNE [2]).

2.4. Antiplane shear motions of a generalized neo-Hookean material. Suppose, from now on, that $\mathcal{R}$ is a cylindrical region and choose a rectangular Cartesian frame $X = \{0; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ so that the unit base vector $\mathbf{e}_3$ is parallel to the generatrix of $\mathcal{R}$. The deformation $\hat{\mathbf{y}}$ defined through (2.1.1) consists of an antiplane shear normal to the plane spanned by the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ if it is of the form

$$\hat{\mathbf{y}}(\mathbf{x},t) = \mathbf{x} + u(x_1, x_2, t)\mathbf{e}_3 \quad \forall(\mathbf{x},t) \in \mathcal{M}. \tag{2.4.1}$$

Observe that the displacement field intrinsic to an antiplane shear deformation of this type has only one nonzero component which lies in the $\mathbf{e}_3$ direction and is independent of the x_3-coordinate. In (2.4.1) $x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha$ for each $\mathbf{x}$ contained in $\mathcal{R}$. The function u will be referred to as the *out-of-plane* displacement field. Inspection of (2.4.1) reveals that any discontinuities in the gradient of $\hat{\mathbf{y}}$ must result from discontinuities in the out-of-plane displacement field and, hence, occur across surfaces which do not vary with the x_3-coordinate. Let S_t denote such a surface at the instant t in $\mathcal{T}$ and let Σ be defined as in (2.1.2).

It is possible to show, following the work of KNOWLES [20] in the inertia-free context, that, although not every hyperelastic, isotropic and incompressible material can sustain nontrivial antiplane shear motions, all generalized neo-Hookean materials are capable of doing so. It is easily shown that for such materials the local balance equations $(2.1.9)_{2,3}$ reduce, in the absense of body forces and under the asssumption that the nominal stress tensor is independent of the x_3-coordinate, to the scalar equation

$$(M(\gamma)u_{,\alpha})_{,\alpha} = \rho\ddot{u} \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \tag{2.4.2}$$

where $\mathcal{X}$ is given by $\mathcal{D} \times \mathcal{T}$, $\mathcal{D}$ is a generic cross section of $\mathcal{R}$, and $\Gamma = \{(x_1, x_2, t) | (x_1, x_2) \in C_t, t \in \mathcal{T}\}$ with $C_t = \mathcal{D} \cap S_t$ at each t in $\mathcal{T}$. See FOSDICK

& SERRIN [13] and FOSDICK & KAO [11] for a general discussion of circumstances under which the field equations $(2.1.9)_{2,3}$ reduce to a single scalar equation. In (2.4.2) M is the secant modulus in shear as defined in (2.2.8) and $\gamma : \mathcal{X} \setminus \Gamma \to \mathbb{R}$ is the *shear strain field* given by

$$\gamma(x_1, x_2, t) = \sqrt{u_{,\alpha}(x_1, x_2, t) u_{,\alpha}(x_1, x_2, t)} \quad \forall (x_1, x_2, t) \in \mathcal{X} \setminus \Gamma. \qquad (2.4.3)$$

For a generalized neo-Hookean material subjected to antiplane shear, the jump condition $(2.1.11)_2$ reduces to

$$[\![M(\gamma) u_{,\alpha}\, n_\alpha]\!] + \rho V_n [\![\dot{u}]\!] = 0 \quad \text{on} \quad \Gamma, \qquad (2.4.4)$$

where $\Gamma = \{(\mathbf{x}, t) | \mathbf{x} \in C_t, t \in \mathcal{T}\}$ and $\mathbf{n}(\cdot, t) : C_t \to \mathcal{N}$ is a unit normal to C_t, while the kinematic jump condition (2.1.14) becomes

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma. \qquad (2.4.5)$$

It is also readily shown that the driving traction f, introduced in Section 2.3, for a generalized neo-Hookean material subjected to an antiplane shear deformation involving a discontinuity in the gradient and, perhaps, the partial derivative with respect to time of out-of-plane displacement field across a moving curve C_t is given, with the aid of (2.3.5), by

$$f = \int_{\bar{\gamma}}^{\overset{+}{\gamma}} \tau(\gamma)\, d\gamma - \langle\!\langle M(\gamma) u_{,\alpha} \rangle\!\rangle [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \qquad (2.4.6)$$

With reference to (2.1.12), (2.1.13) and (2.3.5) it is easily demonstrated that, in the absence of inertial effects, (2.4.2) is replaced by

$$(M(\gamma) u_{,\alpha})_{,\alpha} = 0 \quad \text{on} \quad \mathcal{X} \setminus \Gamma, \qquad (2.4.7)$$

while (2.4.4) becomes

$$[\![M(\gamma) u_{,\alpha} n_\alpha]\!] = 0 \quad \text{on} \quad \Gamma, \tag{2.4.8}$$

and (2.4.6) simplifies to

$$f = \int_{\bar{\gamma}}^{\stackrel{+}{\gamma}} \tau(\gamma)\, d\gamma - M(\stackrel{\pm}{\gamma}) \stackrel{\pm}{u}_{,\alpha} [\![u_{,\alpha}]\!] \quad \text{on} \quad \Gamma. \tag{2.4.9}$$

Observe that, within the context of an antiplane shear deformation of the type described above, no generality is lost by focussing exclusively upon the motion on a cross-section $\mathcal{D}$ of the cylinder $\mathcal{R}$ and the dynamics of the curve $C_t = \mathcal{D} \cap S_t$. In the following, curves C_t across which the gradient and, perhaps, the partial derivative with respect to time of the out-of-plane displacement field $u(\cdot, \cdot, t)$ jump, at some instant t in $\mathcal{T}$, and which segregate the high and low strain phases of the material at hand will, therefore, be referred to as *phase boundaries.*

3. LINEAR STABILITY OF A PROCESS INVOLVING A STEADILY MOVING PLANAR PHASE BOUNDARY IN A THREE-PHASE MATERIAL

3.1. Description of the base process. Suppose that $\mathcal{B}$ is composed of a three-phase material and that the cylinder $\mathcal{R}$ degenerates so as to occupy all of $\mathbb{R}^3$. Let the rectangular Cartesian frame X be as in Section 2.4. Consider an antiplane shear motion on the time interval $(-\infty, 0)$ with an out-of-plane displacement field $u_0(\cdot, t) : \mathbb{R} \to \mathbb{R}$ given by

$$u_0(x_1, t) = \begin{cases} \gamma_l x_1 + v_l t & \text{if} \quad x_1 < v_0 t, \\ \gamma_r x_1 + v_r t & \text{if} \quad x_1 > v_0 t, \end{cases} \tag{3.1.1}$$

for each t in $(-\infty, 0)$, where the shear strains γ_l and γ_r satisfy one of the following:

$$0 < \gamma_r < \underset{*}{\gamma} < \overset{*}{\gamma} < \gamma_l, \quad 0 < \gamma_l < \underset{*}{\gamma} < \overset{*}{\gamma} < \gamma_r. \tag{3.1.2}$$

Since one of (3.1.2) must hold, there is no loss in generality incurred by assuming that the base interface normal velocity v_0 is non-negative; that is,

$$v_0 \geq 0. \tag{3.1.3}$$

It is clear that u_0 satisfies the differential equation in (2.4.2) on the set $(\mathbb{R}^2 \times (-\infty, 0)) \setminus \Gamma_0$ with Γ_0 given by $\{(x_1, x_2, t) | (x_1, x_2) \in A_t, t \in (-\infty, 0)\}$ and $A_t = \{(x_1, x_2) | x_1 = v_0 t, x_2 \in \mathbb{R}\}$ for each t in $(-\infty, 0)$. The moving line A_t is, for each t in $(-\infty, 0)$, a phase boundary.

Assume, in order to comply with the jump conditions in (2.4.4) and (2.4.5) on Γ_0, that the constants γ_l, γ_r, v_l, v_r, and v_0 associated with (3.1.1) are restricted to satisfy the following equations:

$$\begin{aligned} v_r - v_l + v_0(\gamma_r - \gamma_l) = 0, \\ \tau(\gamma_r) - \tau(\gamma_l) + \rho v_0 (v_r - v_l) = 0. \end{aligned} \tag{3.1.4}$$

Assume that the normal velocity of the phase boundary in the base process is *locally subsonic* so that v_0 satisfies the following inequality:

$$v_0 < \min\left\{\sqrt{\tau'(\gamma_l)/\rho}, \sqrt{\tau'(\gamma_r)/\rho}\right\}. \tag{3.1.5}$$

It is then permissible[7] to impose the kinetic relation of the form (2.3.8) or (2.3.9) on Γ_0 and require that the parameters γ_l, γ_r, v_l, v_r, and v_0 satisfy one of

$$v_0 = \tilde{V}(f_0), \quad f_0 = \tilde{\varphi}(v_0), \tag{3.1.6}$$

depending upon whether a kinetic relation of the form (2.3.8) or (2.3.9) is provided. In (3.1.6) the base driving traction f_0 is given, with the aid of (2.4.6), by

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma)\, d\gamma - \tfrac{1}{2}(\tau(\gamma_r) + \tau(\gamma_l))(\gamma_r - \gamma_l). \tag{3.1.7}$$

Observe, as a consequence of (3.1.3) and (2.3.7), that f_0 must satisfy

$$f_0 \geq 0. \tag{3.1.8}$$

In a coordinate frame moving with the phase boundary, the base process described involves a piecewise homogeneous shear strain field. If γ_l and γ_r are consistent with $(3.1.2)_1$ then (3.1.3) implies that the base process is one wherein the high strain elliptic phase of the material at hand *grows* at the expense of the low strain elliptic phase; whereas, if γ_l and γ_r comply with $(3.1.2)_2$ then (3.1.3) implies that the base process is such that the low strain elliptic phase of the material at hand *grows* at the expense of the high strain elliptic phase. In either case the discontinuity involved is, for the duration of the motion, a *normal* phase boundary—that is, the angle between the limiting values of the gradient of the

[7] See ABEYARATNE & KNOWLES [4].

out-of-plane displacement field on either side of the phase boundary is zero at every point of the phase boundary over the time interval $(-\infty, 0)$.

Suppose, in addition to all the above, that the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that its derivative is non-zero at the base driving traction f_0; that is, assume that one of the following—as is appropriate to either (2.3.8) or (2.3.9)—must hold :

$$\tilde{V}'(f_0) \neq 0, \quad \tilde{\varphi}'(v_0) \neq 0. \tag{3.1.9}$$

This assumption is made in order to preclude the necessity of going to higher order in the context of the forthcoming stability analysis. See Figure 2 for the graph of a smooth admissible kinetic reponse function which satisfies (3.1.9).

When inertial effects are ignored it is clear that u_0 as defined in (3.1.1) also satisfies the field equation in (2.4.7) on $(\mathbb{R}^2 \times (-\infty, 0)) \setminus \Gamma_0$. Equation $(3.1.4)_1$ is, in this context, still sufficient to satisfy (2.4.5) on Γ_0. In place of $(3.1.4)_2$, the constants γ_l, γ_r, v_l, v_r, and v_0 must, however, satisfy

$$\tau(\gamma_r) - \tau(\gamma_l) = 0, \tag{3.1.10}$$

in order for the jump condition in (2.4.8) to hold on Γ_0. Although the expression for the base driving traction f_0 given in (3.1.7) remains valid in the inertia-free setting, (3.1.10) can be used to show that

$$f_0 = \int_{\gamma_l}^{\gamma_r} \tau(\gamma)\, d\gamma - \tau_*(\gamma_r - \gamma_l), \tag{3.1.11}$$

where $\tau_* = \tau(\gamma_l) = \tau(\gamma_r)$.

Given a shear stress response function τ which describes a particular three-phase material and an arbitrary kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ which describes the dynamics of phase boundaries which may occur therein, there may or may not, in general, exist constants γ_l, γ_r, v_l, v_r, and v_0 which satisfy one of $(3.1.2)_1$ or $(3.1.2)_2$ and are consistent with the restrictions embodied by (3.1.4), (3.1.5),

$(3.1.6)_1$ or $(3.1.6)_2$, and $(3.1.9)_1$ or $(3.1.9)_2$, or in the inertia-free case, $(3.1.4)_1$, (3.1.10), (3.1.5), $(3.1.6)_1$ or $(3.1.6)_2$, (3.1.11) and $(3.1.9)_1$ or $(3.1.9)_2$. Within the context of this investigation it will be assumed, however, that $\tilde{V}$ or $\tilde{\varphi}$ is chosen so that a non-trivial base process exists.

3.2. Perturbation of the base process. Suppose that at the instant $t = 0$ the out-of-plane displacement and velocity fields and the configuration of the phase boundary associated with the motion specified in Section 3.1 are subjected to a perturbation. Let this perturbation be such that the phase boundary can be, at $t = 0+$, described by the graph C_0 of a continuous function $h : \mathbb{R} \to \mathbb{R}$ of the x_2-coordinate, and segregates elliptic phases of the three-phase material at hand in a sense consistent with that which was present for t in $(-\infty, 0)$. Let the out-of-plane displacement and velocity fields linked to this perturbation be given, respectively, by a once continuously differentiable function $\eta : \mathbb{R}^2 \to \mathbb{R}$ and a continuous function $\varpi : \mathbb{R}^2 \to \mathbb{R}$. Assume that h, η and ϖ represent small deviations, in some appropriate sense, from their counterparts in the base process. In particular, suppose that h, η, $\eta_{,\alpha}$, and ϖ are all square integrable on their domains of definition. Furthermore, require that the components of the gradient of η allow the satisfaction of

$$\lim_{x_1^2+x_2^2\to\infty} \eta_{,\alpha}(x_1,x_2)\eta_{,\alpha}(x_1,x_2) = 0, \tag{3.2.1}$$

while ϖ complies with

$$\lim_{x_1^2+x_2^2\to\infty} \varpi(x_1,x_2) = 0, \tag{3.2.2}$$

so that the disturbance is *localized* in a neighborhood of the phase boundary associated with the base state at $t = 0$.

The perturbation at $t = 0$ will initiate a new process involving an out-of-plane displacement field $u : \mathbb{R}^2 \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ which is, in general, a *weak solution* of the field equation (2.4.2) and satisfies the jump conditions in (2.4.4) and (2.4.5) at all discontinuities in its gradient, the kinetic relation (2.3.8) or (2.3.9) at all

phase boundaries, and the initial conditions

$$
\begin{aligned}
u(\cdot,\cdot,0+) &= u_0(\cdot,0+) + \eta \quad \text{on} \quad \mathbb{R}^2, \\
\dot{u}(\cdot,\cdot,0+) &= \dot{u}_0(\cdot,0+) + \varpi \quad \text{on} \quad \mathbb{R}^2.
\end{aligned}
\tag{3.2.3}
$$

Since the perturbation is small, assume that the subsequent process involves only a single phase boundary $C_t = \{(x_1, x_2, t) |\, x_1 = \varsigma(x_2, t), x_2 \in \mathbb{R}\}$ for each t in $\mathbb{R}_+$, with $\varsigma : \mathbb{R} \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ continuously differentiable on its domain of definition and defined so that it complies with the initial condition

$$
\varsigma(\cdot, 0+) = h \quad \text{on} \quad \mathbb{R}. \tag{3.2.4}
$$

With the intent of linearizing the field equation in (2.4.2) about the base process, write, for each t in $\bar{\mathbb{R}}_+$,

$$
u(x_1, x_2, t) = u_0(x_1, t) + w(x_1, x_2, t) \quad \forall (x_1, x_2) \in \mathcal{D} \setminus C_t, \tag{3.2.5}
$$

where w and its derivatives are assumed to represent small departures from the relevent quantities in the base process. Assume that the components of the gradient of w satisfy the following limits:

$$
\begin{aligned}
\lim_{x_1 \to \pm\infty} w_{,1}(x_1, \cdot, \cdot) &= 0 \quad \text{on} \quad \mathbb{R} \times \bar{\mathbb{R}}_+, \\
\lim_{x_2 \to \pm\infty} w_{,2}(\cdot, x_2, \cdot) &= 0 \quad \text{on} \quad \mathbb{R} \times \bar{\mathbb{R}}_+.
\end{aligned}
\tag{3.2.6}
$$

From (3.2.3) and (3.2.5) it is clear, moreover, that—when inertial effects are not ignored—the increment w to the out-of-plane displacement field must satisfy the following initial conditions:

$$
\begin{aligned}
w(\cdot,\cdot,0+) &= \eta \quad \text{on} \quad \mathbb{R}^2, \\
\dot{w}(\cdot,\cdot,0+) &= \varpi \quad \text{on} \quad \mathbb{R}^2.
\end{aligned}
\tag{3.2.7}
$$

It is important to emphasize that these can not be imposed in the inertia-free setting.

Next, define $s : \mathbb{R} \times \bar{\mathbb{R}}_+ \to \mathbb{R}$, the correction to the interface position due to the perturbation, via

$$\varsigma(\cdot, t) = v_0 t + s(\cdot, t) \quad \text{on} \quad \mathbb{R} \quad \forall t \in \bar{\mathbb{R}}_+. \tag{3.2.8}$$

Note, from (3.2.4) that the increment s to the phase boundary position must satisfy the initial condition

$$s(\cdot, 0+) = h \quad \text{on} \quad \mathbb{R}. \tag{3.2.9}$$

Observe that the unit normal vectors $\mathbf{n}_\pm(\cdot, t) : \mathbb{R} \to \mathcal{N}$ to C_t are given by

$$\mathbf{n}_\pm(\cdot, t) = \pm \frac{\mathbf{e}_1 - s_{,2}(\cdot, t)\mathbf{e}_2}{\sqrt{1 + s_{,2}^2(\cdot, t)}} \quad \text{on} \quad \mathbb{R} \quad \forall t \in \mathbb{R}_+. \tag{3.2.10}$$

For the remainder of this work, choose the unit normal vector associated with the plus sign in (3.2.10) and drop this sign when referring to it. The normal velocity $V_n(\cdot, t) : \mathbb{R} \to \mathbb{R}$ of C_t is given, for each t in $\mathbb{R}_+$, by

$$V_n(\cdot, t) = \frac{v_0 + \dot{s}(\cdot, t)}{\sqrt{1 + s_{,2}^2(\cdot, t)}} \quad \text{on} \quad \mathbb{R} \quad \forall t \in \mathbb{R}_+. \tag{3.2.11}$$

3.3. Linearization of the field equations associated with the process initiated by the perturbation. Let $\mathcal{D}_t^l$ and $\mathcal{D}_t^r$ denote, for each t in $\mathbb{R}_+$, plane sets defined by

$$\mathcal{D}_t^l = \{(x_1, x_2) | \, x_1 \le \varsigma(x_2, t)\}, \quad \mathcal{D}_t^r = \mathbb{R}^2 \setminus \mathring{\mathcal{D}}_t^l. \tag{3.3.1}$$

Let $\mathcal{X}_l$ and $\mathcal{X}_r$ be given, in turn, by

$$\mathcal{X}_l = \{(x_1, x_2, t) | \, (x_1, x_2) \in \mathcal{D}_t^l, t \in \mathbb{R}_+\}, \tag{3.3.2}$$

and

$$\mathcal{X}_r = \{(x_1, x_2, t) | (x_1, x_2) \in \mathcal{D}_t^r, t \in I\!R_+\}. \tag{3.3.3}$$

The field equations which hold on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$ can be obtained by linearizing the partial differential equation in (2.4.2) about γ_l and γ_r, respectively. First, consider the derivation of the field equation which holds on $\mathring{\mathcal{X}}_l$. From (2.4.3), (3.2.5) and the assumption regarding the magnitude of the spatial gradient of the increment w to the out-of-plane displacement field it is clear that

$$\begin{aligned} \gamma &= \sqrt{(\gamma_l + w_{,1})^2 + w_{,2}^2} = \sqrt{\gamma_l^2 + 2\gamma_l w_{,1} + w_{,\alpha} w_{,\alpha}} \\ &\cong \gamma_l + w_{,1} \quad \text{on} \quad \mathring{\mathcal{X}}_l. \end{aligned} \tag{3.3.4}$$

From (2.2.8), (3.3.4) and Taylor's theorem it is further evident that

$$M(\gamma) \cong M(\gamma_l + w_{,1}) \cong M(\gamma_l) + M'(\gamma_l) w_{,1} \quad \text{on} \quad \mathring{\mathcal{X}}_l. \tag{3.3.5}$$

Next, using (3.2.5) and (3.3.5) in the left-hand-side of the partial differential equation in (2.4.2) gives

$$\begin{aligned} (M(\gamma) u_{,\alpha})_{,\alpha} &\cong [(M(\gamma_l) + M'(\gamma_l) w_{,1}) u_{,\alpha}]_{,\alpha} \\ &\cong [(M(\gamma_l) + M'(\gamma_l) w_{,1})(\gamma_l + w_{,1})]_{,1} \\ &\qquad + [(M(\gamma_l) + M'(\gamma_l) w_{,1}) w_{,2}]_{,2} \\ &\cong \tau'(\gamma_l) w_{,11} + M(\gamma_l) w_{,22} \quad \text{on} \quad \mathring{\mathcal{X}}_l \end{aligned} \tag{3.3.6}$$

Note that, in deriving (3.3.6), the smoothness of τ and hence M, the identity

$$\tau(\gamma) = M(\gamma)\gamma \quad \forall \gamma \in \bar{I\!R}_+,$$

which follows from (2.2.6) and (2.2.8) and its consequence

$$\tau'(\gamma) = M(\gamma) + M'(\gamma)\gamma \quad \forall \gamma \in \bar{I\!R}_+ \setminus \{\underset{*}{\gamma}, \overset{*}{\gamma}\}$$

have been used. Observe, also, that $\tau'(\gamma_l) > 0$ by whichever of (3.1.2) is appropriate, and $M(\gamma_l) > 0$ by (2.2.9). From (3.1.1), (3.2.5) and (3.3.6), the linearized field equation which holds on $\mathring{\mathcal{X}}_l$ is

$$a_l^2 w_{,11} + b_l^2 w_{,22} = \ddot{w}, \tag{3.3.7}$$

where the positive constants a_l and b_l are defined by

$$a_l = \sqrt{\tau'(\gamma_l)/\rho}, \quad b_l = \sqrt{M(\gamma_l)/\rho}. \tag{3.3.8}$$

Similarly, the linearized field equation which holds on $\mathring{\mathcal{X}}_r$ is

$$a_r^2 w_{,11} + b_r^2 w_{,22} = \ddot{w}, \tag{3.3.9}$$

where the positive constants a_r and b_r are defined by

$$a_r = \sqrt{\tau'(\gamma_r)/\rho}, \quad b_r = \sqrt{M(\gamma_r)/\rho}. \tag{3.3.10}$$

In writing (3.3.10), the positivity of $\tau'(\gamma_r) > 0$ and $M(\gamma_r) > 0$, which are results of (2.2.9) and whichever of $(3.1.2)_{1,2}$ is appropriate, have been used.

From (2.4.7) and (3.3.6) it is clear that, in the inertia-free setting, equations (3.3.7) and (3.3.9) are supplanted by

$$a_l^2 w_{,11} + b_l^2 w_{,22} = 0, \tag{3.3.11}$$

and

$$a_r^2 w_{,11} + b_r^2 w_{,22} = 0, \tag{3.3.12}$$

which hold on $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$, repectively.

3.4. Linearization of the jump conditions and kinetic relation associated with the process initiated by the perturbation. Since the set

$\Gamma = \{(x_1, x_2, t) | (x_1, x_2) \in C_t, t \in I\!R_+\}$ represents the post-disturbance trajectory of the phase boundary, the jump conditions in (2.4.4) and (2.4.5) and the kinetic balance equation in (2.3.8) or (2.3.9)—with V_n and f given, respectively, by (3.2.11) and (2.4.6)—must hold on it. Assume, henceforth, that the function s introduced via (3.2.8) and its derivatives are small in the same sense that w is small. Note, first, that this assumption implies, using (3.2.10) and (3.2.11), the following approximations for $\mathbf{n}$ and V_n on Γ:

$$\mathbf{n} \cong \mathbf{e}_1 - s_{,2}\,\mathbf{e}_2 \quad \text{on} \quad \Gamma, \quad V_n \cong v_0 + \dot{s} \quad \text{on} \quad \Gamma. \tag{3.4.1}$$

It will now be shown that a further consequence of the above stipulation regarding the size of s and its derivatives is that, within the error associated with the linearization, the jump conditions in (2.4.4) and the kinetic relation can be enforced on an undisturbed continuation of the phase boundary intrinsic to the base process I given by

$$I = \{(x_1, x_2, t) | x_1 = v_0 t, x_2 \in I\!R, t \in I\!R_+\}, \tag{3.4.2}$$

but, when V_n appears in any of these, the contribution due to $\dot{s}$ from $(3.4.1)_2$ must be retained. To see this consider, for example, the limiting values of the x_1–component of the gradient of the out-of-plane displacement field u on either side of the phase boundary; note that by Taylor's theorem, (3.1.1), and (3.2.8),

$$\begin{aligned} u_{,1}(\varsigma(x_2,t)-, x_2, t) &:= \lim_{h \searrow 0} u_{,1}(v_0 t - h + s(x_2,t), x_2 + h s_{,2}(x_2,t), t) \\ &\cong \gamma_l + w_{,1}((v_0 t + s(x_2,t))-, x_2, t) \\ &\cong \gamma_l + w_{,1}(v_0 t-, x_2, t) + w_{,11}(v_0 t-, x_2, t) s(x_2,t) \\ &= u_{,1}(v_0 t-, x_2, t) + w_{,11}(v_0 t-, x_2, t) s(x_2,t) \\ &\qquad \forall (x_2, t) \in I\!R \times I\!R_+, \end{aligned} \tag{3.4.3}$$

and, similarly,

$$\begin{aligned} u_{,1}(\varsigma(x_2,t)+,x_2,t) &:= \lim_{h\searrow 0} u_{,1}(v_0t+h+s(x_2,t),x_2+hs_{,2}(x_2,t),t) \\ &\cong \gamma_r + w_{,1}((v_0t+s(x_2,t))+,x_2,t) \\ &\cong \gamma_r + w_{,1}(v_0t+,x_2,t) + w_{,11}(v_0t+,x_2,t)s(x_2,t) \\ &= u_{,1}(v_0t+,x_2,t) + w_{,11}(v_0t+,x_2,t)s(x_2,t) \\ &\qquad \forall(x_2,t)\in \mathbb{R}\times\mathbb{R}_+. \end{aligned} \tag{3.4.4}$$

Since both s and the derivatives of w are assumed small and of the same order, the quadratic terms on the right-hand-sides of (3.4.3) and (3.4.4) can be neglected. This produces

$$u_{,1}(\varsigma(x_2,t)\pm,x_2,t) \cong u_{,1}(v_0t\pm,x_2,t) \quad \forall(x_2,t)\in \mathbb{R}\times\mathbb{R}_+. \tag{3.4.5}$$

Within the scope of the linearization, $u_{,1}(\varsigma(x_2,t)\pm,x_2,t)$ is, therefore, obtained, for each (x_2,t) in $\mathbb{R}\times\mathbb{R}_+$, by evaluating $u_{,1}(\cdot,x_2,t)$ at $v_0t\pm$. Analogous remarks also hold for the x_2–component of the gradient of the out-of-plane displacement field, the out-of-plane velocity field and the shear strain field. Hence, with the aid of (3.4.1), (2.4.4) implies that

$$\begin{aligned} 0 =& \tau(\gamma_r) - \tau(\gamma_l) + \rho v_0(v_r - v_l) + \rho v_0(\gamma_r - \gamma_l)\dot{s}(x_2,t) \\ &+ \rho\left(a_r^2 w_{,1}(v_0t+,x_2,t) - a_l^2 w_{,1}(v_0t-,x_2,t)\right) \\ &+ \rho v_0\left(\dot{w}(v_0t+,x_2,t) - \dot{w}(v_0t-,x_2,t)\right) \quad \forall(x_2,t)\in \mathbb{R}\times\mathbb{R}_+. \end{aligned} \tag{3.4.6}$$

From $(3.1.4)_2$, the constant term on the right-hand-side of (3.4.6) is zero and, hence, the linearization of the jump condition which enforces the balance of linear momentum across the phase boundary leads to

$$\begin{aligned} 0 =& a_r^2 w_{,1}(v_0t+,\cdot,t) - a_l^2 w_{,1}(v_0t-,x_2,t) + v_0(\gamma_r - \gamma_l)\dot{s}(x_2,t) \\ &+ v_0\left(\dot{w}(v_0t+,x_2,t) - \dot{w}(v_0t-,x_2,t)\right) \quad \forall(x_2,t)\in \mathbb{R}\times\mathbb{R}_+. \end{aligned} \tag{3.4.7}$$

Prior to deriving the linearized kinetic relation it is convenient linearize the driving traction f. From (3.2.4), (2.4.3) and the foregoing discussion one finds that

$$\begin{aligned} f(x_2,t) \cong f_0 &+ \tfrac{1}{2}\rho(\gamma_l-\gamma_r)(a_r^2 w_{,1}(v_0t+,x_2,t) + a_l^2 w_{,1}(v_0t-,x_2,t)) \\ &+ \tfrac{1}{2}(\tau(\gamma_r)-\tau(\gamma_l))(w_{,1}(v_0t+,x_2,t) - w_{,1}(v_0t-,x_2,t)) \\ &\forall (x_2,t) \in I\!R \times I\!R_+, \end{aligned} \tag{3.4.8}$$

where the base driving traction f_0 is given by (3.1.7). From $(3.1.4)_2$ it is clear, furthermore, that (3.4.8) simplifies to read

$$\begin{aligned} f(x_2,t) \cong f_0 + \tfrac{1}{2}\rho(\gamma_l-\gamma_r)((a_r^2-v_0^2)w_{,1}(v_0t+,x_2,t) + (a_l^2-v_0^2)w_{,1}(v_0t-,x_2,t)) \\ \forall (x_2,t) \in I\!R \times I\!R_+. \end{aligned} \tag{3.4.9}$$

If the kinetic relation is of the form given in (2.3.8), then $(3.4.1)_2$, (3.4.9) and Taylor's theorem lead to

$$\begin{aligned} v_0 + \dot{s}(x_2,t) &= \tilde{V}(f_0) \\ &+ \tfrac{1}{2}\rho\tilde{V}'(f_0)(\gamma_l-\gamma_r)((a_r^2-v_0^2)w_{,1}(v_0t+,x_2,t) + (a_l^2-v_0^2)w_{,1}(v_0t-,x_2,t)) \\ &\forall (x_2,t) \in I\!R \times I\!R_+. \end{aligned} \tag{3.4.10}$$

If, on the other hand, the kinetic relation is provided in the form (2.3.9), then $(3.4.1)_2$, (3.4.9) and Taylor's theorem give, similarly,

$$\begin{aligned} \tilde{\varphi}(v_0) + \tilde{\varphi}'(v_0)\dot{s}(x_2,t) &= f_0 \\ &+ \tfrac{1}{2}\rho(\gamma_l-\gamma_r)((a_r^2-v_0^2)w_{,1}(v_0t+,x_2,t) + (a_l^2-v_0^2)w_{,1}(v_0t-,x_2,t)) \\ &\forall (x_2,t) \in I\!R \times I\!R_+. \end{aligned} \tag{3.4.11}$$

Use of the $(3.1.6)_1$ and $(3.1.6)_2$ in (3.4.10) and (3.4.11), respectively, results in the linearized kinetic relation

$$\dot{s}(x_2,t) = \frac{\gamma_l - \gamma_r}{2v_*}((a_r^2 - v_0^2)w_{,1}(v_0t+,x_2,t) + (a_l^2 - v_0^2)w_{,1}(v_0t-,x_2,t)) \quad \forall (x_2,t) \in I\!R \times I\!R_+, \tag{3.4.12}$$

where the constant v_* is defined by either

$$v_* = \frac{1}{\rho \tilde{V}'(f_0)}, \tag{3.4.13}$$

if the kinetic relation is furnished in the form (2.3.8), or

$$v_* = \frac{\tilde{\varphi}'(v_0)}{\rho}, \tag{3.4.14}$$

if the kinetic relation is supplied in the form (2.3.9). Note, from (3.1.9), that v_* is a real—but nonzero—constant.

Consider, now, the task of linearizing the jump condition in (2.4.5). Note, from (3.2.5), that (2.4.5) implies

$$0 = u_0(\zeta(x_2,t)+,t) - u_0(\zeta(x_2,t)-,t) + w(\zeta(x_2,t)+,x_2,t) - w(\zeta(x_2,t)-,x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+, \tag{3.4.15}$$

Certainly, from (3.1.1), (3.2.8) and $(3.1.4)_1$,

$$u_0(\zeta(x_2,t)+,t) - u_0(\zeta(x_2,t)-,t) = (\gamma_l - \gamma_r)s(x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+; \tag{3.4.16}$$

furthermore, from (3.2.8), Taylor's theorem, and the assumption regarding the small magnitude of products involving s and the derivatives of w,

$$\begin{aligned} w(\varsigma(x_2,t)\pm,x_2,t) &\cong w(v_0t\pm,x_2,t) + w_{,1}(v_0t\pm,x_2,t)s(x_2,t) \\ &\cong w(v_0t\pm,x_2,t) \quad \forall (x_2,t) \in I\!R \times I\!R_+. \end{aligned} \tag{3.4.17}$$

Hence, from (3.4.15)-(3.4.17), the linearization of the jump condition which enforces the continuity of displacement across the phase boundary yields

$$w(v_0t+,x_2,t) - w(v_0t-,x_2,t) = (\gamma_l - \gamma_r)s(x_2,t) \quad \forall(x_2,t) \in I\!R \times I\!R_+. \quad (3.4.18)$$

Differentiation of (3.4.18) with respect to time then results in the following identity:

$$\begin{aligned} 0 =& \dot{w}(v_0t+,x_2,t) - \dot{w}(v_0t-,x_2,t) + (\gamma_r - \gamma_l)\dot{s}(x_2,t) \\ &+ v_0(w_{,1}(v_0t+,x_2,t) - w_{,1}(v_0t-,x_2,t)) \\ &\qquad \forall(x_2,t) \in I\!R \times I\!R_+, \end{aligned} \quad (3.4.19)$$

Appropriate substitution of (3.4.19) into the linearization of the jump condition which enforces the balance of linear momentum across the phase boundary (3.4.7) then gives rise to

$$\begin{aligned} 0 =& (a_r^2 - v_0^2)w_{,1}(v_0t+,x_2,t) - (a_l^2 - v_0^2)w_{,1}(v_0t-,x_2,t) \\ &+ 2v_0(\gamma_l - \gamma_r)\dot{s}(x_2,t) \quad \forall(x_2,t) \in I\!R \times I\!R_+. \end{aligned} \quad (3.4.20)$$

By virtue of the foregoing calculations it is crucial to note that, within the scope of the linearization, it is legitimate to enforce the partial differential equations in (3.3.7) and (3.3.9) on the interiors of the sets Ω_l and Ω_r defined by

$$\Omega_l = \{(x_1,x_2,t) | (x_1,x_2) \in \Pi_t^l, t \in I\!R_+\}, \quad (3.4.21)$$

with $\Pi_t^l = \{(x_1,x_2) | x_1 \leq v_o t, x_2 \in I\!R\}$ for each t in $I\!R_+$, and

$$\Omega_r = \{(x_1,x_2,t) | (x_1,x_2) \in \Pi_t^r, t \in I\!R_+\}, \quad (3.4.22)$$

with $\Pi_t^r = \{(x_1,x_2) | x_1 \geq v_o t, x_2 \in I\!R\}$ for each t in $I\!R_+$, instead of the sets $\mathring{\mathcal{X}}_l$ and $\mathring{\mathcal{X}}_r$ (recall (3.3.2) and (3.3.3)).

In the inertia-free case it is readily shown that, while (3.4.18) continues to hold, (3.4.17) is replaced by

$$a_r^2 w_{,1}(v_0 t+, x_2, t) - a_l^2 w_{,1}(v_0 t-, x_2, t) = 0 \quad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.4.23}$$

and (3.4.11) simplifies to read

$$\dot{s}(x_2, t) = \frac{\gamma_l - \gamma_r}{2v_*}\left(a_r^2 w_{,1}(v_0 t+, x_2, t) + a_l^2 w_{,1}(v_0 t-, x_2, t)\right) \qquad \forall (x_2, t) \in \mathbb{R} \times \mathbb{R}_+. \tag{3.4.24}$$

Finally, remarks analogous to those made regarding the enforcement of the partial differential equations in (3.3.7) and (3.3.9) on $\mathring{\Omega}_l$ and $\mathring{\Omega}_r$ apply also to those in (3.3.11) and (3.3.12).

3.5. Linearized description of the post perturbation process. In this section the linearized field equations, jump conditions, kinetic relation, initial conditions (where appropriate), and far field decay conditions satisfied by the increments w and s to the out-of-plane displacement field and the interface position are listed in both the inertial and inertia-free cases.

In the inertial case, (3.3.7) and (3.3.9) give the following linearized field equations

$$\begin{aligned} a_l^2 w_{,11} + b_l^2 w_{,22} &= \ddot{w} \quad \text{on} \quad \mathring{\Omega}_l, \\ a_r^2 w_{,11} + b_r^2 w_{,22} &= \ddot{w} \quad \text{on} \quad \mathring{\Omega}_r. \end{aligned} \tag{3.5.1}$$

In addition, from (3.4.20) and (3.4.18), the following jump conditions hold

$$\begin{aligned} [\![(a^2 - v_0^2) w_{,1}]\!] &= 2v_0(\gamma_r - \gamma_l)\dot{s} \quad \text{on} \quad I, \\ [\![w]\!] &= (\gamma_l - \gamma_r)s \quad \text{on} \quad I, \end{aligned} \tag{3.5.2}$$

where

$$a_+^2 = a_r^2, \quad a_-^2 = a_l^2. \tag{3.5.3}$$

Next, from (3.4.12) the following linearized kinetic relation holds:

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle (a^2 - v_0^2) w_{,1} \rangle\!\rangle \quad \text{on} \quad I. \tag{3.5.4}$$

The initial conditions satisfied by w and s are, from (3.2.4) and (3.2.7),

$$\begin{aligned} w(\cdot,\cdot,0+) &= \eta \quad \text{on} \quad \mathbb{R}^2, \\ \dot{w}(\cdot,\cdot,0+) &= \varpi \quad \text{on} \quad \mathbb{R}^2, \\ s(\cdot,0+) &= h \quad \text{on} \quad \mathbb{R}. \end{aligned} \tag{3.5.5}$$

Finally, from (3.2.6), it is assumed that the following far field conditions hold

$$\begin{aligned} \lim_{x_1 \to \pm\infty} w_{,1}(x_1,\cdot,t) &= 0 \quad \text{on} \quad \mathbb{R}, \\ \lim_{x_2 \to \pm\infty} w_{,2}(\cdot,x_2,t) &= 0 \quad \text{on} \quad \mathbb{R}, \end{aligned} \tag{3.5.6}$$

for each t in $\mathbb{R}_+$.

In the inertia-free case, (3.5.1) is replaced by the following:

$$\begin{aligned} a_l^2 w_{,11} + b_l^2 w_{,22} &= 0 \quad \text{on} \quad \mathring{\Omega}_l, \\ a_r^2 w_{,11} + b_r^2 w_{,22} &= 0 \quad \text{on} \quad \mathring{\Omega}_r. \end{aligned} \tag{3.5.7}$$

Furthermore, the jump condition $(3.5.2)_1$ is, by virtue of (3.4.23), replaced by

$$[\![a^2 w_{,1}]\!] = 0 \quad \text{on} \quad I, \tag{3.5.8}$$

while $(3.5.2)_2$ continues to hold. Following (3.4.24), the linearized kinetic relation (3.5.4) is superseded by

$$\dot{s} = \frac{\gamma_l - \gamma_r}{v_*} \langle\!\langle a^2 w_{,1} \rangle\!\rangle \quad \text{on} \quad I, \tag{3.5.9}$$

In the absence of inertial effects initial conditions cannot be given for the increments to the out-of-plane displacement and velocity fields w and $\dot{w}$; the initial

condition $(3.5.5)_3$ pertaining to s still, however, continues to be applicable. The decay conditions (3.5.6) also still hold.

3.6. Normal mode analysis in the inertia-free setting. An approximate means for analyzing the linear stability of the base process described in Section 3.1 is afforded by the study of the inertia-free initial value problem consisting of (3.5.7)–(3.5.9), $(3.5.2)_2$, $(3.5.5)_3$ and (3.5.6). Observe that, by virtue of the linearization, the relevant partial differential equations, jump conditions and kinetic relation are all linear with constant coefficients; note, also, that the domains $\mathring{\Pi}_t^l$ and $\mathring{\Pi}_t^r$ are, for each t in $\mathbb{R}_+$, rectangular. It is therefore possible to find a solution to the linearized partial differential equations, jump conditions and kinetic relation in the form

$$w(x_1,x_2,t) = \begin{cases} W_l e^{+\xi_l(x_1-v_0t)} e^{i\kappa x_2} e^{pt} & \forall (x_1,x_2) \in \mathring{\Pi}_t^l, \quad t \in \mathbb{R}_+, \\ W_r e^{-\xi_r(x_1-v_0t)} e^{i\kappa x_2} e^{pt} & \forall (x_1,x_2) \in \mathring{\Pi}_t^r, \quad t \in \mathbb{R}_+, \end{cases} \tag{3.6.1}$$

$$s(x_2,t) = S e^{i\kappa x_2} e^{pt} \quad \forall (x_2,t) \in \mathbb{R} \times \mathbb{R}_+,$$

where the *amplitudes* W_l, W_r and S, *wave-numbers* ξ_l, ξ_r and κ, and *growth-rate* p are all constants. To comply with the decay condition $(3.5.6)_1$ it is clear that $\Re(\xi_l)$ and $\Re(\xi_r)$ must both be positive. The *Ansatz* (3.6.1) is not, in general, consistent with the initial condition $(3.5.5)_3$ or the decay condition $(3.5.6)_2$; since the initial disturbance h is stipulated to be square integrable on $\mathbb{R}$, and hence can be represented as a Fourier integral—

$$h(x_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{h}(\kappa) e^{i\kappa x_2}\, d\kappa \quad \forall x_2 \in \mathbb{R}, \tag{3.6.2}$$

it is reasonable to expect that stability results can be obtained by a *normal-mode analysis*; such an analysis entails substitution of (3.6.1) into (3.5.7)–(3.5.9) to determine the growth-rate p as a function of the positive wave-numbers ξ_l and ξ_r and the real wave-number κ. In the context of such an undertaking, the

amplitude S and wave-number κ are regarded as given and non-zero, while, along with the growth-rate p, the amplitudes W_l and W_r and wave-numbers ξ_l and ξ_r are—due to the present lack of inertial effects—to be determined. If there exists a complex growth-rate p with positive real part which arises as a solution to the aforementioned problem then the base process will be referred to as *linearly unstable*. Otherwise, the base process will be called *linearly stable*.

Substitution of (3.6.1) into (3.5.7)-(3.5.9) and $(3.5.2)_2$ yields the following system of five equations in the five unknowns W_l, W_r, ξ_l, ξ_r and p:

$$
\begin{aligned}
(a_l^2\xi_l^2 - b_l^2\kappa^2)W_l &= 0,\\
(a_r^2\xi_l^2 - b_r^2\kappa^2)W_r &= 0,\\
a_r^2\xi_r W_r + a_l^2\xi_l W_l &= 0,\\
W_r - W_l - (\gamma_r - \gamma_l)S &= 0,\\
\frac{\gamma_l - \gamma_r}{2v_*}(a_r^2\xi_r W_r - a_l^2\xi_l W_l) + Sp &= 0.
\end{aligned}
\tag{3.6.3}
$$

First, $(3.6.3)_{1,2}$ give, recalling that ξ_l and ξ_r must be positive,

$$
\xi_l = \frac{b_l}{a_l}|\kappa|, \quad \xi_r = \frac{b_r}{a_r}|\kappa|. \tag{3.6.4}
$$

In particular, (3.6.4) implies that, the wave-numbers ξ_l and ξ_r must be real and, further, that, of ξ_l, ξ_r and κ, the growth-rate p will depend only on κ. Next, $(3.6.3)_{3,4,5}$ can be solved for the remaining unknowns W_l, W_r and p to yield

$$
\begin{aligned}
W_l &= -\frac{\nu^2|\kappa|}{a_l b_l(\gamma_l - \gamma_r)}S,\\
W_r &= -\frac{\nu^2|\kappa|}{a_r b_r(\gamma_r - \gamma_l)}S,\\
p &= -\frac{\nu^2}{v_*}|\kappa|,
\end{aligned}
\tag{3.6.5}
$$

where the constant ν^2 is defined as follows:

$$
\nu^2 = \frac{a_l b_l a_r b_r(\gamma_l - \gamma_r)^2}{a_l b_l + a_r b_r}. \tag{3.6.6}
$$

Equation $(3.6.5)_3$ which gives the growth-rate p in terms of the wave-number κ and various physical parameters intrinsic to the problem at hand will be referred to as the *dispersion relation.* It is clear from $(3.6.5)_3$ that the growth-rate p is real. Since $\kappa \neq 0$ by assumption and $v_* \neq 0$ by (3.1.10), it is clear, moreover, that the signs of p and v_* are opposite: *viz.*,

$$\operatorname{sgn}(p) = \operatorname{sgn}(-v_*). \tag{3.6.7}$$

The linear stability of the base process, in the absence of inertial effects, thus depends entirely upon the sign of v_*. Significantly, the wave-number κ plays no role in determining stability. Moreover, the local mechanical properties of the high and low strain elliptic phases of the three-phase material at hand do not effect the stability criteria. If $v_* > 0$ then the base process is linearly stable with respect to all initial disturbances h of the type under consideration. If, alternatively, $v_* < 0$ then the base process is linearly unstable with respect to all disturbances h of the type considered here. To summarize, in the absence of inertial effects, the criterion $v_* < 0$ is necessary and sufficient for the base process to be unstable with respect to any perturbation of the type put into consideration in Section 3.2. Thus, if the kinetic response function $\tilde{V}$ or $\tilde{\varphi}$—as is appropriate to whether a kinetic relation of the form (2.3.8) or (2.3.9) is prescribed—is non-monotonic, the base process may be linearly unstable with respect to any initial disturbance h of the type being considered. Discussion regarding the physical reasonableness of a non-monotonic kinetic response function is left until Section 3.9.

Suppose, now, that the normal velocity v_0 of the phase boundary in the base process is zero. Note that this is equivalent to requiring that the base process be mechanically equilibrated. Then, by the admissibility of the kinetic response function, one or both of f_0 and v_* must be zero. Since the latter contradicts (3.1.10), $v_0 = 0$ implies, at present, that $f_0 = 0$ and, furthermore, either $\tilde{V}'(0) > 0$ or $\tilde{\varphi}'(0) > 0$. Thus the foregoing stability dichotomy implies that a mechanically equilibrated base process of the type defined in Section 3.1 must be linearly stable

with respect to all initial disturbances h of the type considered here.

Note that the foregoing results are consistent with those presented by FRIED [14] in a study of the distribution of driving traction along a particular non-planar phase boundary. The latter work shows, roughly speaking, that the driving traction is less in regions of the interface where the curvature is larger. Specifically, if small deviations of the relevant type from a planar interface with constant base driving traction f_* are considered and a kinetic response function $\tilde{V}$ is provided so that $\tilde{V}'(f_*) < 0$ then the normal velocity of the interface in regions of higher curvature exceeds that in regions of lesser curvature. The interface, then, has a marked tendency to evolve in a manner wherein its curved portions move ahead of its flat portions—and, hence, become less planar. Such a response is intuitively *unstable*. If, however, f_* is such that $\tilde{V}'(f_*) > 0$, flat portions of the interface tend to *catch up* with the curved portions so that the interface regains its planar shape—hence, the planar interface is *stable*.

3.7. An alternative to normal mode analysis in the inertia-free setting. The analysis performed in Section 3.6 resulted in a necesary and sufficient condition for the base process to be unstable with respect to an arbitrary perturbation of the type discussed in Section 3.2; it did not, however, encompass a means for *tracking* the evolution of either a stable or unstable response to perturbation. This section will focus on an analysis which does allow the post-perturbation evolution of the phase boundary to be followed in the linear regime. Since the analysis is performed in the linear realm it is, in the case of an unstable response, only of value in a short time interval following the perturbation at $t = 0$. Consider, again, the inertia-free initial boundary value problem consisting of (3.5.7)-(3.5.9), $(3.5.2)_2$, $(3.5.5)_3$ and (3.5.6). Techniques from potential theory are used in the appendix of this work to show that the increment $w(\cdot, \cdot, t)$ to the out-of-plane displacement field can be represented, for each t in $\mathbb{R}_+$, in terms of the sum of a single-layer potential, with density proportional to $\dot{s}$, and a double-layer potential potential, with density proportional to s—each on the line

$x_1 = v_0 t$—as follows:

$$w(x_1,x_2,t) = \int_{-\infty}^{+\infty} K_1(x_1 - v_0 t, x_2 - \zeta, \tfrac{b_d}{a_d})\dot{s}(\zeta,t)\,d\zeta$$

$$+ \int_{-\infty}^{+\infty} K_2(x_1 - v_0 t, x_2 - \zeta, \tfrac{b_d}{a_d}) s(\zeta,t)\,d\zeta \quad \forall (x_1,x_2) \in \mathring{\Pi}_t^d. \tag{3.7.1}$$

The sub- and superscripts d in (3.7.1) are to be replaced by either l or r—as appropriate, while the kernels $K_\alpha(\cdot,\cdot,c) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ are defined via

$$\begin{aligned} K_1(x,y,c) &= \frac{1}{2\pi}\frac{v_*}{\nu^2(\gamma_l - \gamma_r)} \ln\sqrt{c^2x^2+y^2} \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ K_2(x,y,c) &= \frac{1}{2\pi}\frac{cx}{c^2x^2+y^2} \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \end{aligned} \tag{3.7.2}$$

for each c in $\mathbb{R}_+$, and s, introduced in (3.2.8), satisfies the following functional initial value problem

$$\begin{aligned} \dot{s}(x_2,t) &= \frac{\nu^2}{\pi v_*} \,⨍_{-\infty}^{+\infty} \frac{s_{,2}(\zeta,t)\,d\zeta}{x_2-\zeta} \quad \forall (x_2,t) \in \mathbb{R} \times \mathbb{R}_+, \\ s(x_2,0) &= h(x_2) \quad \forall x_2 \in \mathbb{R}. \end{aligned} \tag{3.7.3}$$

The integral on the right-hand-side of $(3.7.3)_1$ is, as indicated, of the Cauchy principal value type.

The stability of the base process rests on the stability of the solution s to the foregoing functional initial value problem. To investigate this issue assume—because of the square integrability of h—that s can be represented in the form

$$s(x_2,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{s}(\kappa,t)e^{i\kappa x_2}\,d\kappa \quad \forall (x_2,t) \in \mathbb{R} \times \bar{\mathbb{R}}_+, \tag{3.7.4}$$

for some function $\hat{s} : \mathbb{R} \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ which satisfies

$$\hat{s}(\kappa,0) = \int_{-\infty}^{+\infty} h(x_2)e^{-i\kappa x_2}\,dx_2 =: \hat{h}(\kappa) \quad \forall \kappa \in \mathbb{R}. \tag{3.7.5}$$

Now, from (3.7.3)–(3.7.5) it is possible to derive, for each fixed κ in $\mathbb{R}$, the following initial value problem:

$$\begin{aligned} \dot{\hat{s}}(\kappa,\cdot) + \frac{\nu^2}{v_*}|\kappa|\hat{s}(\kappa,\cdot) = 0 \quad \text{on} \quad \mathbb{R}, \\ \hat{s}(\kappa,0) = \hat{h}(\kappa). \end{aligned} \tag{3.7.6}$$

Observe that the Laplace transform could be applied to (3.7.6) to derive the dispersion relation $(3.6.5)_3$. The linear stability results obtained in Section 3.6 would, then, follow immediately. Instead, note that (3.7.6) can be solved, formally, to yield the following expression for $\hat{s}$:

$$\hat{s}(\kappa,t) = \hat{h}(\kappa)e^{-\frac{\nu^2}{v_*}|\kappa|t} \quad \forall(\kappa,t) \in \mathbb{R} \times \bar{\mathbb{R}}_+. \tag{3.7.7}$$

Inspection of (3.7.7) reveals that, as demonstrated in Section 3.6, the linear stability of the base process is decided entirely by the sign of v_*. Note that if $v_* > 0$ then, by (3.7.4) and (3.7.7),

$$s(x_2,t) = \frac{\nu^2 t}{\pi v_*} \int_{-\infty}^{+\infty} \frac{h(\zeta)\,d\zeta}{(x_2-\zeta)^2 + (\frac{\nu^2 t}{v_*})^2} \quad \forall(x_2,t) \in \mathbb{R} \times \bar{\mathbb{R}}_+. \tag{3.7.8}$$

Hence, under these circumstances, it is clear that

$$\lim_{t\to\infty} s(x_2,t) = 0 \quad \forall x_2 \in \mathbb{R}, \tag{3.7.9}$$

for all square integrable initial disturbances h. The base process is, therefore, linearly stable with respect to all such disturbances.

Suppose, now, that $v_* < 0$—which, as remarked in Section 3.6, can occur only if $v_0 > 0$. It is then instructive to consider a special example where the initial disturbance h is given by

$$h(x_2) = \frac{1}{\pi}\frac{A}{1+(\frac{x_2}{\ell})^2} \quad \forall x_2 \in \mathbb{R}, \tag{3.7.10}$$

with ℓ taken, without loss of generality, to be a positive constant. See Figure 2 for a representative graph of h. Note that since $(3.7.3)_1$ is linear it is, in general, possible to consider an initial disturbance which involves a linear superposition of disturbances of the type defined in (3.7.10), *e.g.*, $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(x_2) = \frac{1}{\pi} \sum_{n=1}^{N} \frac{A_n}{1 + (\frac{x_2 - y_n}{\ell_n})^2} \quad \forall x_2 \in \mathbb{R},$$

where N is a natural number and A_n, y_n and ℓ_n are real constants for each n in $\{1, 2, \ldots, N\}$. The results which follow are easily extended to apply to such a generalization of the initial disturbance. Now, the Fourier transform $\hat{h}$ of the function h defined in (3.7.10) is given by

$$\hat{h}(\kappa) = A\ell e^{-\ell|\kappa|} \quad \forall \kappa \in \mathbb{R}. \tag{3.7.11}$$

Next, (3.7.4), (3.7.7) and (3.7.11) imply that

$$s(x_2, t) = \frac{A\ell}{2\pi} \int_{-\infty}^{+\infty} e^{-(\ell - \frac{\nu^2 t}{|v_*|})|\kappa|} e^{i\kappa x_2} \, d\kappa \quad \forall (x_2, t) \in \mathbb{R} \times [0, t_c), \tag{3.7.12}$$

where the *critical* time t_c is chosen small enough so that the amplitude of s does not contradict the assumptions necessary for the linearization to remain valid over the time interval $[0, t_c)$; *i.e.*,

$$t_c \ll \frac{|v_*|\ell}{\nu^2} =: t_\infty. \tag{3.7.13}$$

From (3.7.12) it transpires, further, that

$$s(x_2, t) = \frac{1}{\pi} \frac{A t_\infty (t_\infty - t)}{(t_\infty - t)^2 + t_\infty^2 (\frac{x_2}{\ell})^2} \quad \forall (x_2, t) \in \mathbb{R} \times [0, t_c). \tag{3.7.14}$$

The expression for $s(\cdot, t)$ on $\mathbb{R}$ given in (3.7.14) shows that the amplitude and wavelength of the hump associated with the special initial disturbance under

consideration increase and decrease, respectively, as t approaches t_c. Hence, up until some critical time, the assumption $v_* < 0$ implies that the hump associated with the initial disturbance grows in an unstable fashion. See Figure 4 for a depiction of this unstable evolution. Observe, moreover, that, as t approaches t_∞, the function s given by (3.7.14) forms a *delta sequence* so that

$$\lim_{t \to t_\infty} s(\cdot, t) = A\ell\delta \quad \text{on} \quad \mathbb{R}, \tag{3.7.15}$$

where δ is the Dirac delta distribution.

3.8. Normal mode and energy analysis with inertial effects included. The applicability of the stability criterion obtained in Section 3.6, and recovered in Section 3.7, is, so far, limited to contexts where inertial effects are insignificant. This section is concerned with the extension, as far as possible, of the inertia-free stability dichotomy based upon the sign of v_* into the inertial realm. As a first step toward achieving this goal a normal mode analysis analogous to that performed in Section 3.6 is undertaken with inertial terms present. The conclusion of this analysis is that $v_* < 0$ is necessary and sufficient to guarantee the linear instability of the base process with respect to a particular subset of the class of perturbations put into consideration in Section 3.2. It is then argued that the remaining class of perturbations not covered by the normal mode analysis is, in fact, very small. Next an energy argument is used to show that $v_* < 0$ is a necessary condition for the base process to be linearly unstable with respect to an arbitrary disturbance within the full class of perturbations introduced in Section 3.2.

Consider, now, the initial boundary value problem composed by (3.5.1), (3.5.2) and (3.5.4)-(3.5.6). It is, once again, possible to find a solution to the system formed by the linearized field equations, jump conditions and kinetic relation in the form of (3.6.1). Remarks regarding the decomposition of the initial data—which now includes η and ϖ as well as h—and the satisfaction of $(3.5.6)_2$

akin to those made in Section 3.6 are pertinent. Observe that, in the inertial case, the amplitudes W_r, W_l and S and wave-numbers ξ_l, ξ_r and κ must be viewed as given for the normal mode analysis to be effective in determining necessary and sufficient conditions—via a dispersion relation analogous to $(3.6.5)_3$—for instability with respect to arbitrary disturbances of the out-of-plane displacement and velocity fields and the interface within the class of perturbations put forth in Section 3.2. Substitution, however, of (3.6.1) into (3.5.1), (3.5.2) and (3.5.4) produces the following relations

$$
\begin{aligned}
\xi_l &= \frac{f_l(\kappa,p) - v_0 p}{a_l^2 - v_0^2}, \\
\xi_r &= \frac{f_r(\kappa,p) + v_0 p}{a_r^2 - v_0^2}, \\
W_l &= \frac{(\gamma_l - \gamma_r)(v_0 p - f_r(\kappa,p))}{f_l(\kappa,p) + f_r(\kappa,p)} S, \\
W_r &= \frac{(\gamma_l - \gamma_r)(v_0 p + f_l(\kappa,p))}{f_l(\kappa,p) + f_r(\kappa,p)} S, \\
p &= -\frac{(\gamma_l - \gamma_r)^2 (f_l(\kappa,p) f_r(\kappa,p) + v_0^2 p^2)}{v_*(f_l(\kappa,p) + f_r(\kappa,p))},
\end{aligned}
\tag{3.8.1}
$$

where $f_l : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ and $f_r : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are given by

$$
\begin{aligned}
f_l(\kappa,p) &= \sqrt{(a_l^2 - v_0^2) b_l^2 \kappa^2 + a_l^2 p^2} \quad \forall (\kappa,p) \in \mathbb{R} \times \mathbb{C}, \\
f_r(\kappa,p) &= \sqrt{(a_r^2 - v_0^2) b_r^2 \kappa^2 + a_r^2 p^2} \quad \forall (\kappa,p) \in \mathbb{R} \times \mathbb{C}.
\end{aligned}
\tag{3.8.2}
$$

The square roots in (3.8.2) are defined so that when their respective arguments are real and positive the resulting square root is also real and positive.

It is clear from $(3.8.1)_{1,2,3,4}$ that, for (3.6.1) to represent a solution to (3.5.1), (3.5.2) and (3.5.4), the amplitudes W_l and W_r and wave-numbers ξ_l and ξ_r cannot be chosen independently of S and κ. Hence, the normal mode procedure is only capable of analyzing the linear stability of the base process with respect to a certain class of perturbations; that is, it is only possible—via this analysis—to

determine conditions necessary and sufficent for the instability of the base process with respect to a subset of the class of perturbations put into consideration in Section 3.2. To do this it suffices to analyze the zero structure of the dispersion relation $(3.8.1)_5$ as a function of the growth-rate p for fixed values of the wave-number κ and the parameters γ_l, γ_r, v_0, a_l, a_r, b_l, b_r, ρ and v_*. This is done below.

If $v_* < 0$ it is evident, by inspection, that there exists a real positive root p to $(3.8.1)_5$ for all admissible values of κ, γ_l, γ_r, v_0, a_l, a_r, b_l, b_r and ρ. Hence, $v_* < 0$ is sufficient—indepedent of the value of the wave-number κ—to guarantee that the base process is unstable with respect to the narrowed class of initial disturbances at hand. To establish the converse, show that the condition $v_* > 0$ is sufficent to guarantee that there cannot exist unstable zeros p to $(3.8.1)_5$. Let $F(\kappa, \cdot) : \mathbb{C} \to \mathbb{C}$ be given, for each κ in $\mathbb{R}$, by

$$F(\kappa, p) = \frac{(\gamma_l - \gamma_r)^2 (f_l(p,\kappa) f_r(p,\kappa) + v_0^2 p^2)}{f_l(p,\kappa) + f_r(p,\kappa)} \quad \forall p \in \mathbb{C}. \tag{3.8.3}$$

Now, it is easy to show that if $\Re(p) > 0$ then $\Re(F(\kappa, p)) > 0$ for every κ in $\mathbb{R}$. Hence, if $v_* > 0$ then all roots p to $(3.8.1)_5$ must have non-positive real parts. Thus, $v_* < 0$ is also a necessary condition for the base process to be unstable with respect to an arbitrary element of the class of perturbations which can be tested via the normal mode analysis. Observe that this conclusion is independent of the wave-number κ associated with (3.6.1).

If, in place of the foregoing normal mode analysis, a full-fledged Fourier-Laplace transform analysis of (3.5.1), (3.5.2) and (3.5.4)–(3.5.6) is performed, then the narrowing of the class of initial data necessitated by the normal mode analysis does not occur. Furthermore, in this case it transpires that the Fourier-Laplace transform of s can be expressed in the form

$$S(\kappa, p) = \frac{\hat{h}(\kappa) + \frac{\gamma_l - \gamma_r}{v_*} H(\kappa, p)}{p + \frac{1}{v_*} F(\kappa, p)} \quad \forall (\kappa, p) \in \mathbb{R} \times \mathbb{C}, \tag{3.8.4}$$

where $\hat{h}$ is the Fourier transform of h and, for each (κ, p) in $I\!R \times \mathbb{C}$, $H(\kappa, p)$ is a functional of the initial data η and ϖ. Evidently, it is possible that there exist combinations of η, ϖ and h which would allow the cancellation of an unstable zero in the denominator of the expression on the right-hand-side of (3.8.4) by a zero in its numerator. It is equally clear, however, that the set of such initial data constitutes a very small one within the full class of initial data purveyed in Section 3.2. Hence, except in response to very special initial perturbations the condition $v_* < 0$ is necessary and sufficent for the base state to be linearly unstable. The normal mode analysis thus shows that $v_* < 0$ is necessary and sufficient for the base process to be unstable with respect to all but a very small subset of the initial disturbances under consideration.

Now, to show that $v_* < 0$ is a necessary condition for the base state to be linearly unstable with respect to any perturbation in the full class introduced in Section 3.2 consider the dependence of the following energy $\mathcal{E}$ on time:

$$\begin{aligned}\mathcal{E}(t) =& \frac{\rho\ell}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{0}\left(\dot{v}^2(x_1,x_2,t) + (a_l^2 - v_0^2)v_{,1}^2(x_1,x_2,t) + b_l^2 v_{,2}^2(x_1,x_2,t)\right)dx_1dx_2 \\ &+ \frac{\rho\ell}{2}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\left(\dot{v}^2(x_1,x_2,t) + (a_r^2 - v_0^2)v_{,1}^2(x_1,x_2,t) + b_r^2 v_{,2}^2(x_1,x_2,t)\right)dx_1dx_2 \\ & \qquad \forall t \in [0, t_*). \qquad (3.8.5)\end{aligned}$$

In (3.8.5) ℓ is a positive constant which carries units of length and the function $v : I\!R^2 \times \bar{I\!R}_+ \to I\!R$ is defined via

$$w(x_1, x_2, t) = v(x_1 - v_0 t, x_2, t) \quad \forall (x_1, x_2, t) \in I\!R^2 \times \bar{I\!R}_+. \qquad (3.8.6)$$

By $(3.5.5)_{1,2}$ and the stipulated square integrability of $\eta_{,\alpha}$ and ϖ it is evident that $\mathcal{E}(0)$ exists. Assume, then, that there exists a positive time t_*, which may be very small, such that the integrals which define $\mathcal{E}$ exist on the time interal

$(0, t_*)$. Recalling (3.1.5), (3.3.8) and (3.3.10), it is clear that $\mathcal{E}$ is non-negative on its domain of definition. Now, an alternate definition of linear stability in terms of the energy $\mathcal{E}$ is that it remain bounded on $\bar{\mathbb{R}}_+$. A series of calculations which use (3.5.2) and (3.5.4)–(3.5.6) then show that the *power* $\dot{\mathcal{E}}$ is given by

$$\dot{\mathcal{E}}(t) = -\rho \ell v_* \int_{-\infty}^{+\infty} \dot{s}^2(x_2, t)\, dx_2 \quad \forall t \in [0, t_*). \tag{3.8.7}$$

Certainly, if $v_* > 0$ then $\dot{\mathcal{E}}(t) \leq 0$ for all t in $[0, t_*)$; furthermore, under these circumstances, the interval over which $\mathcal{E}$ is defined can be extended in increments to $\bar{\mathbb{R}}_+$ leading to the following inequality

$$\dot{\mathcal{E}}(t) \leq 0 \quad \forall t \in \bar{\mathbb{R}}_+. \tag{3.8.8}$$

The condition $v_* > 0$ is, therefore, sufficient to ensure that the energy $\mathcal{E}$ remains bounded for all time and, hence—according to the above definition of linear stability—that the base process is stable with respect to all perturbations of the type introduced in Section 3.2. Now, since $v_* \neq 0$ by assumption, it is clear that $v* < 0$ is a necessary condition for the base process to become unstable with respect to any perturbation of the type under consideration.

From the foregoing discussion it is clear that admissibility, the assumed smoothness of $\tilde{V}$ or $\tilde{\varphi}$ and (3.1.9) imply that when $v_0 = 0$, the base driving traction $f_0 = 0$ and $\tilde{V}'(0) > 0$ or $\tilde{\varphi}'(0) > 0$ and, hence, a mechanically equilibrated base process of the kind introduced in Section 3.1 is—as in the inertia-free case—stable with respect to all perturbations considered in this work.

To recapitulate, the calculations performed this section show that $v_* < 0$ is a necessary condition for the base process to be unstable with respect to any initial disturbance of the type under consideration and a sufficient condition for the base process to be unstable to all but a small subset of these initial disturbances. Comparing the results obtained in Section 3.6 with those obtained here, it is apparent that inertial effects do not significantly effect the linear stability criteria.

3.9. Discussion. The analysis of Sections 3.6 and 3.8 shows that—regardless of whether intertial effects are included or not—$v_* < 0$ is necessary for the base process to be linearly unstable with respect to any perturbation of the type purveyed in Section 3.2 and sufficient for the base process to be unstable with respect to all but a small subset of these perturbations. In Section 3.7 it is shown, in the absence of inertial effects, that if $v_* < 0$ then the linear instability will manifest itself in a manner whereby the morphology of the interface evolves so as to develop plate-like or dendritic structures. Recall, from the alternate definitions (3.4.13) and (3.4.14) of v_*, that $v_* < 0$ can occur only if the relevant kinetic response function $\tilde{V}$ or $\tilde{\varphi}$ is locally decreasing at f_0 or v_0, respectively. Is it physically plausible for $\tilde{V}$ or $\tilde{\varphi}$ to display such non-monotonicity? Recall from Section 2.3 that admissibility—from the perspective of the Clausius-Duhem inequality—does not restrict the monotonicity of the kinetic response function. OWEN, SCHOEN & SRINIVASAN [26] suggest, moreover, that *unstable kinetics* of the sort where $\tilde{\varphi}$ has a single maximum as a function of V_n—and thus must be a non-monotonic function of V_n—may be responsible for the rapid growth of plate-like structures which is observed experimentally. Furthermore, there exist other physical contexts, the most notable of which include unstable crack growth and the *slip-stick* peeling of tape,[8] where the analogues of such non-monotonic kinetic response functions are considered physically acceptable.

It is reasonable, based on the foregoing discussion, to refer to the type of linear instability which occurs when $v_* < 0$ as *kinetic instablity.* This investigation has demonstrated that, under the current kinematical and constitutive restrictions, in a purely mechanical context, independent of whether inertial effects are accounted for or not, the only means by which a linear instability involving the emergence of plate-like or dendritic structures from a planar interface can occur is if a kinetic instability is present. It is possible that, in a broader context, other brands of instability may be present. This may, in particular, be true when thermal effects

[8] See AIFANTIS [7], AUBREY, WELDING & WONG [8] and MAUGIS & BARQUINS [23].

are taken into consideration. An investigation which takes both mechanical and thermal effects into consideration is performed by FRIED [15].

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APPENDIX

In this appendix (3.7.1) and (3.7.3) are established. Consider the inertia-free initial value problem comprised of (3.5.7)–(3.5.9), $(3.5.2)_2$, $(3.5.5)_3$ and (3.5.6). Define $v : I\!R^2 \times I\!R_+ \to I\!R$ by

$$w(x_1, x_2, t) = v(\hat{\eta}_1(x_1, t), \hat{\eta}_2(x_2), t) \quad \forall (x_1, x_2, t) \in I\!R^2 \times I\!R_+, \tag{A.1}$$

where $\hat{\eta}_1(\cdot, t) : I\!R \to I\!R$ and $\hat{\eta}_2 : I\!R \to I\!R$ are given by

$$\hat{\eta}_1(x_1, t) = \begin{cases} \frac{b_l}{a_l}(x_1 - v_0 t) & \text{if} \quad x_1 < v_0 t, \\ \frac{b_r}{a_r}(x_1 - v_0 t) & \text{if} \quad x_1 > v_0 t, \end{cases} \tag{A.2}$$

for each t in $\bar{I\!R}_+$, and

$$\hat{\eta}(x_2) = x_2 \quad \forall x_2 \in I\!R. \tag{A.3}$$

Then, in terms of v, (3.5.7)–(3.5.9) yield, for each t in $I\!R_+$,

$$\begin{aligned} v_{,\alpha\alpha} &= 0 \quad \text{on} \quad ((-\infty, 0) \cup (0, \infty)) \times I\!R, \\ a_r b_r v_{,1}(0+, \cdot, t) - a_l b_l v_{,1}(0-, \cdot, t) &= 0 \quad \text{on} \quad I\!R, \\ v(0+, \cdot, t) - v(0-, \cdot, t) &= (\gamma_l - \gamma_r) s(\cdot, t) \quad \text{on} \quad I\!R \\ a_r b_r v_{,1}(0+, \cdot, t) + a_l b_l v_{,1}(0-, \cdot, t) &= \frac{2 v_*}{(\gamma_l - \gamma_r)} \dot{s}(\cdot, t) \quad \text{on} \quad I\!R. \end{aligned} \tag{A.4}$$

Now, from $(A.4)_{2,4}$ it is clear that

$$v_{,1}(0+, \cdot, t) - v_{,1}(0-, \cdot, t) = \frac{v_*(a_l b_l - a_r b_r)}{a_l b_l a_r b_r (\gamma_l - \gamma_r)} \dot{s}(\cdot, t) \quad \text{on} \quad I\!R, \tag{A.5}$$

for each t in $I\!R_+$ Consider, next, $(A.4)_{1,3}$ and (A.5). Since $v(\cdot, \cdot, t)$ is harmonic on $((-\infty, 0) \cup (0, \infty)) \times I\!R$ for each t in $\bar{I\!R}_+$, the jumps in the normal derivative of v and in v itself across the line $l = \{(\eta_1, \eta_2) | \, \eta_1 = 0, \eta_2 \in I\!R\}$ indicate that it

can be represented by the sum of appropriate single- and double-layer potentials with densities proportional to $\dot{s}$ and s, respectively. That is, for each t in $\mathbb{R}_+$,

$$v(\eta_1, \eta_2, t) = \int_{-\infty}^{+\infty} \left(L_1(\eta_1, \eta_2 - \zeta)\dot{s}(\zeta, t) + L_2(\eta_1, \eta_2 - \zeta)s(\zeta, t) \right) d\zeta$$
$$\forall (\eta_1, \eta_2) \in ((-\infty, 0) \cup (0, \infty)) \times \mathbb{R}, \qquad \text{(A.6)}$$

where $L_\alpha : \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R}_+ \to \mathbb{R}$ are given by

$$\begin{aligned} L_1(\eta_1, \eta_2) &= \frac{1}{2\pi} \frac{v_*}{\nu^2(\gamma_l - \gamma_r)} \ln \sqrt{\eta_1^2 + \eta_2^2} \quad \forall (\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ L_2(\eta_1, \eta_2) &= \frac{1}{2\pi} \frac{\eta_1}{\eta_1^2 + \eta_2^2} \quad \forall (\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0,0)\}, \end{aligned} \qquad \text{(A.7)}$$

and where ν^2 is as defined in (3.6.6). With the aid of (A.1)-(A.3), (A.6) gives the representation (3.7.1) of w. Next, use of standard identities from potential theory gives the following expression for the limiting values of $v_{,1}(\cdot, \cdot, t)$ on either side of l

$$v_{,1}(0\pm, \eta_2, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{s_{,2}(\zeta, t)\, d\zeta}{\eta_2 - \zeta} \pm \frac{v_*(a_l b_l - a_r b_r)}{2 a_l b_l a_r b_r (\gamma_l - \gamma_r)^2} \dot{s}(\eta_2, t) \quad \forall \eta_2 \in \mathbb{R}, \qquad \text{(A.8)}$$

for each t in $\mathbb{R}_+$. Substitution of (A.8) into either $(\text{A.4})_2$ or $(\text{A.4})_4$ then yields the functional equation $(3.7.3)_1$. The initial condition $(3.7.3)_2$ follows directly from $(3.5.5.)_3$.

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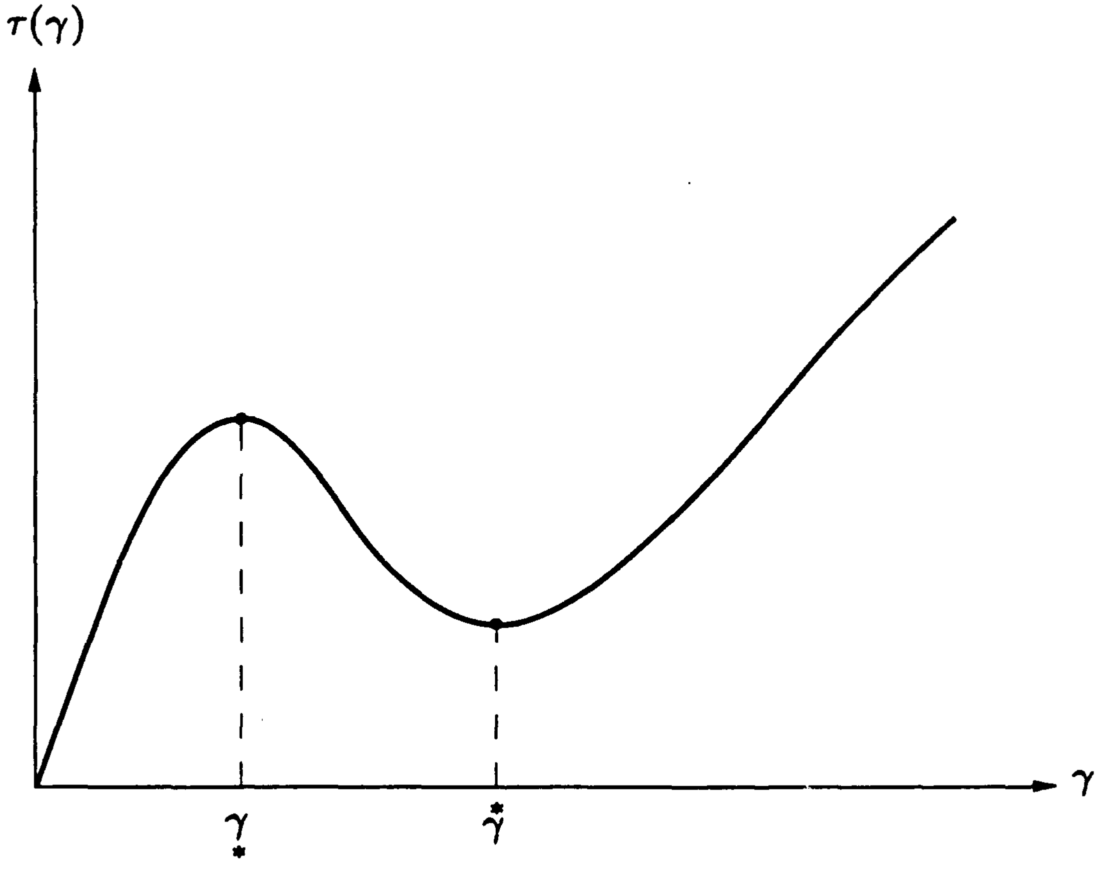

Figure 1: Graph of the shear stress response function τ.

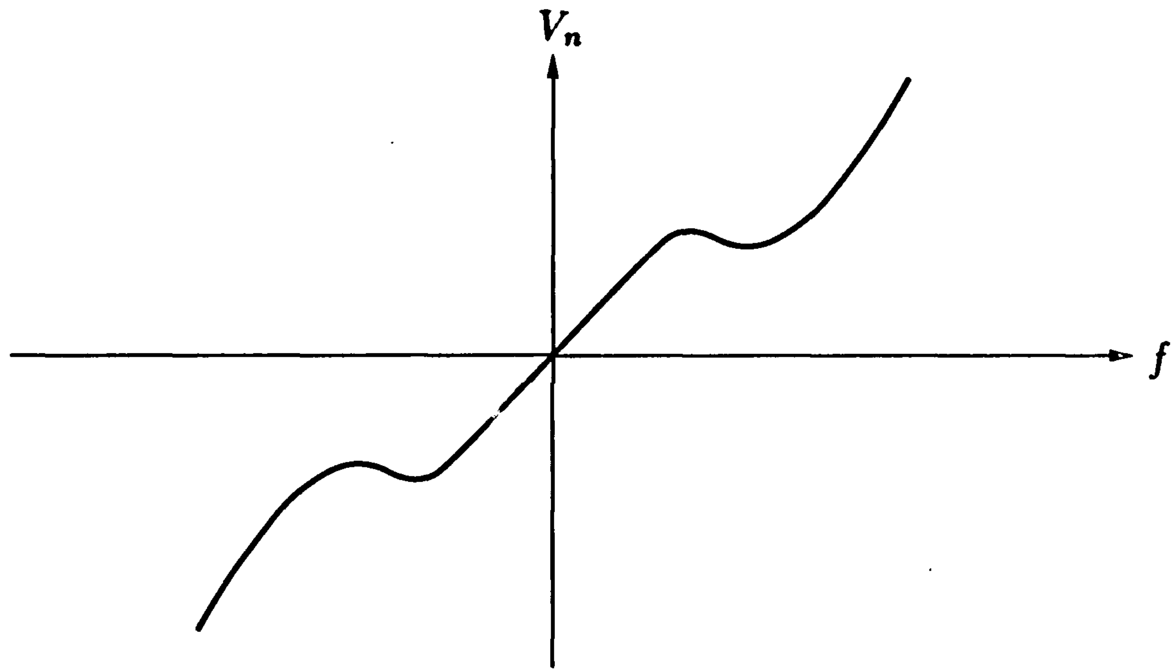

Figure 2: Graph of the kinetic response function $\tilde{V}$.

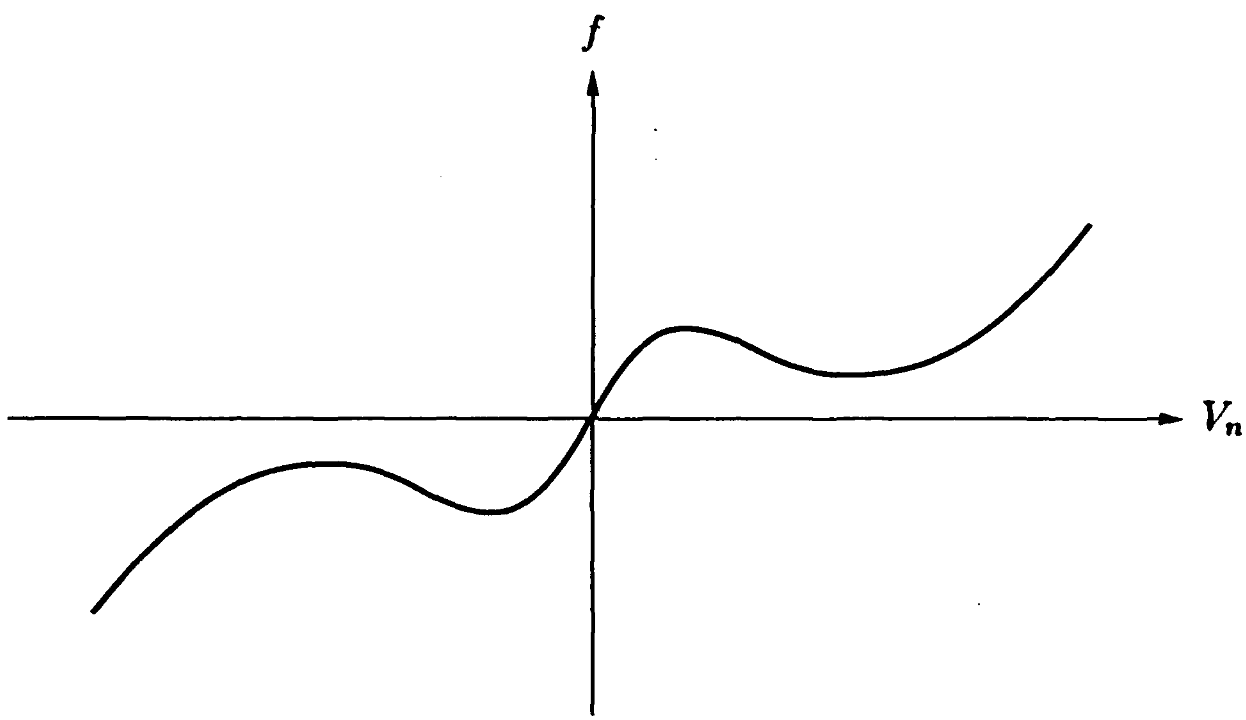

Figure 3: Graph of the kinetic response function $\tilde{\varphi}$.

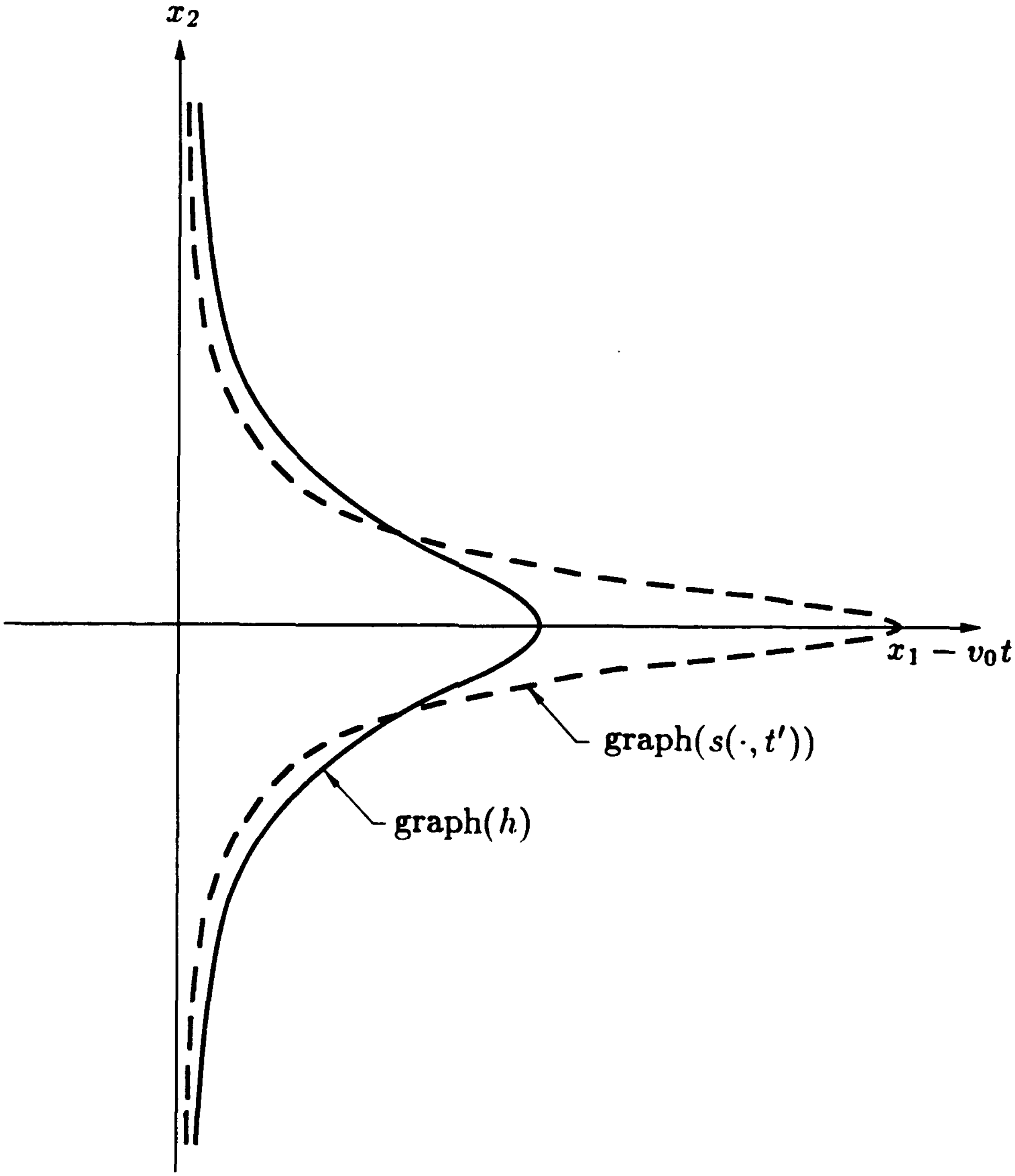

Figure 4: Superimposed graphs of the h and $s(\cdot, t')$ for some $t' \ll t_c$.

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This work investigates the linear stability of an antiplane shear motion which involves a planar phase boundary in an arbitrary element of a wide class on non-elliptic generalized neo-Hookean materials which have two distinct elliptic phases. It is shown, via a normal mode analysis, that, in the absence of inertial effects, such a process is linearly unstable with respect to a large class of disturbances if and only if the kinetic response function -- a constitutively supplied entity which gives the normal velocity of a phase boundary in terms of the driving traction which acts on it or vice versa -- is locally decreasing as a function of the appropriate argument. An alternate analysis, in which the linear stability problem is recast as a functional equation for the interface position, allows the interface to be tracked subsequent to perturbation. A particular choice of the initial disturbance is used to show that, in the case of an unstable response, the morphological character of the phase boundary evolves to qualitatively resemble the plate-like structures which are found in displacive solid-solid

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#19. phase transformations. In the presence of inertial effects a combination of normal mode and energy analyses are used to show that the condition which is necessary and sufficient for instability with respect to the relevant class of perturbations and sufficient for all but a very special, and physically unrealistic, subclass of these perturbations. The linear stability of the relevant process depends, therefore, entirely upon the transformation kinetics intrinsic to the kinetic response function.